EASTERN UNIVERSITY, SRI LANKA

FACULTY OF SCIENCE

First Year Second Semester Examination in Science - 2021/2022

(August/September, 2024)

MT 1032 - Limit Process

Answer all questions | Time allowed: Two hours

Q1. (a) i. Define what is meant by an inductive set in $\mathbb{R}$. [10 Marks]

ii. Prove that the set $\mathbb{N}$ is the smallest inductive set in $\mathbb{R}$. [15 Marks]

(b) i. Define the terms '*Supremum*' and '*Infimum*' of a non-empty subset of $\mathbb{R}$. [10 Marks]

ii. Prove that a lower bound l of a non-empty bounded below subset S of $\mathbb{R}$ is the infimum of S if and only if for every $\epsilon > 0$, there exists $x \in S$ such that $x < l + \epsilon$. [25 Marks]

(c) Suppose that A and B are nonempty bounded above subsets of $\mathbb{R}$. Define

$$A + B = \{a + b /\ a \in A,\ b \in B\}.$$

i. Prove that $A + B$ is bounded above.

ii. Prove that $\sup(A + B) \leq \sup(A) + \sup(B)$. [40 Marks]

Q2. (a) State what it means by a sequence of real numbers (x_n) converges to a limit a. [10 Marks]

Use the definition of convergence of sequence of real numbers to show that

$$\lim_{n\to\infty} \frac{2n^2 + 3n - 5}{5n^2 + 2n + 1} = \frac{2}{5}.$$

[20 Marks]

(b) Suppose that (x_n) is a sequence of real numbers and that $a, b \in \mathbb{R}$ with $a \neq 0$. If (x_n) converges to x, show that $(|ax_n + b|)$ converges to $|ax + b|$. [20 Marks]

(c) State the Monotone Convergent Theorem. [10 Marks]

Let $x_1 = 1$ and $x_{n+1} = \frac{1}{4}(2x_n + 3)$ for $n \in \mathbb{N}$. Prove that

i. (x_n) is a monotonically strictly increasing sequence.

ii. $x_n < 2$ for all $n \in \mathbb{N}$.

iii. $\lim\limits_{n\longrightarrow\infty} (x_n) = \frac{3}{2}$. [40 Marks]

Q3. (a) Define the following terms:

i. a subsequence of a sequence;

ii. Cauchy sequence. [20 Marks]

(b) State the Bolzano-Weierstrass Theorem and use it to prove that the sequence (x_n), where $x_n = \dfrac{(n^2+20n+35)\sin n^3}{n^2+n+1}$, has a convergent subsequence. [25 Marks]

(c) Let (x_n) be a Cauchy sequence. Use the definition to prove that the sequence $(2x_n + 5)$ is also a Cauchy sequence. [25 Marks]

(d) Prove that, if (x_n) is Cauchy and has a subsequence (x_{n_k}) which is convergent, then (x_n) is also convergent, and converges to the same limit. [30 Marks]

Q4. (a) Define what is meant by the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent. [10 Marks]

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

converges or diverges. If it is convergent find its sum. [20 Marks

(b) Define the following terms:

i. Absolute convergence; [10 Marks

ii. Conditional convergence. [10 Marks]

Determine whether the following series converges absolutely, converges conditionally or diverges. Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$$

[15 Mark

(c) Find the values of x for the power series $\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+1)4^n}$, which converges.

[35 Mark