EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2018/219

SECOND SEMESTER (July, 2022)

MT 3253 - GROUP THEORY-I

Answer all questions

Time : Three hours

1. (a) Define the term *group*.

   (b) Show that the cancelation laws hold in a group.

   (c) In a group G, prove the following:

      i. $(a^{-1})^{-1} = a$ for all $a \in G$.

      ii. $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$.

   (d) If $a^2 = e$ for all elements a in a group G, then show that G is abelian.

2. (a) A nonempty subset H of a group G is a subgroup of G if and only if $a, b \in H$ implies that $ab^{-1} \in H$ for every $a, b \in H$.

   (b) Let H be a subgroup of a group G. Prove that the identity element of H is the same of the identity element of G.

   (c) Let a be a fixed element of a group of G. The *centralizer* of a in G is

$$C(a) = \{g \in G \mid ga = ag\}.$$

   Prove the following:

      i. $C(a)$ is a subgroup of G,

      ii. $C(a) = C(a^{-1})$.

3. (a) Define the term *right coset* of a subgroup H in a group G.

   (b) Let H be a subgroup of a group G and $g_1, g_2 \in G$. Prove the following:

      i. $Hg = H$ if and only if $g \in H$,

      ii. $Hg_1 = Hg_2$ if and only if $g_1g_2^{-1} \in H$.

   (c) Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $h \in H$ and $g \in G$. Show that every left coset gH is the same as the right coset Hg.

4. (a) Define the term *homomorphism* between two groups.
   (b) Let $\phi : G \to G'$ be a homomorphism of a group G into a group G'. Prove the following:
      i. if e is the identity element of G, then $\phi(e)$ is the identity element of G',
      ii. if $g \in G$, then $\phi(g^{-1}) = (\phi(g))^{-1}$,
      iii. if G is abelian, then $\phi(G)$ is abelian.
      iv. if G is cyclic, then $\phi(G)$ is cyclic.
   (c) Let $G = \mathbb{R}$ under addition and let $H = \mathbb{R}^+$ under multiplication, and let $\phi : G \to H$ be a mapping defined by $\phi(x) = e^x$. Then show that ϕ is a homomorphism.

5. (a) Define the term *normal subgroup* of a group.
   (b) Show that the intersection of two normal subgroups is a normal subgroup.
   (c) If H is a subgroup of G and K is a normal subgroup of G, then show that $H \cap K$ is a normal subgroup of H.
   (d) If N and M are normal subgroups of a group G, show that

$$NM = \{nm \mid n \in N, m \in M\}$$

   is also a normal subgroup of G.

6. (a) Define the term *factor subgroup* of a group.
   (b) Prove that a factor group of a cyclic group is cyclic.
   (c) If H is a subgroup of an abelian group G, then show that the factor group G/H must be abelian.
   (d) Let $N = \langle 6 \rangle = \{0, 6, 12\}$ be a normal subgroup of $G = \mathbb{Z}_{18}$. Find the elements of the factor group $G/\mathbb{Z}$.