EASTERN UNIVERSITY, SRI LANKA

FACULTY OF SCIENCE

Third Year First Semester Examination in Science - 2021/2022

(Sept./Oct., 2024)

MT 3013 - COMPLEX ANALYSIS I

Answer all questions **Time: Three hours**

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1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$. [Marks 05]

   (b) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ-neighborhood of a point $z_0 = x_0 + iy_0$. Suppose that the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood and that they are continuous at (x_0, y_0). Prove that, if those partial derivatives satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

   at $z_0 = x_0 + iy_0$, then the derivative $f'(z_0)$ exists. [Marks 60]

   (c) Define what is meant by the function $h : \mathbb{R}^2 \to \mathbb{R}$ being **harmonic**. [Marks 05]

   Find the harmonic conjugate of $e^{-3x}\cos(3y)$. [Marks 30]

2. (a) i. Define what is meant by a **path** $\gamma : [\alpha, \beta] \to \mathbb{C}$.

   ii. For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)\, dz$.

   [Marks 10]

   (b) Let $D(a; r) := \{z \in \mathbb{C} : |z - a| < r\}$ denotes the open disc with center $a \in \mathbb{C}$ and radius $r > 0$ and let f be analytic on $D(a; r)$ and $0 < s < r$. Prove **Cauchy's Integral Formula**,

$$f(z_0) = \frac{1}{2\pi i}\int_{C(a;s)} \frac{f(z)}{z - z_0}\, dz, \quad \text{for } z_0 \in D(a; s),$$

   where $C(a; s)$ denotes the circle with center a and radius $s > 0$. [Marks 40]

(c) Let $C(0;1)$ be the unit circle $z = e^{it}$, $0 \leqslant t \leqslant 2\pi$. Find the integral

$$\int_{C(0;1)} \frac{dz}{(z-2)(z-\frac{1}{2})}.$$

Hence find that

$$\int_0^{2\pi} \frac{dt}{5 - 4\cos t}.$$

[Mark

3. (a) State the **Mean-Value Property for Analytic Function.** [Marks

(b) i. Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being **entire**. [Marks

ii. Prove **Liouville's Theorem**: If f is entire and

$$\frac{\max\{|f(t)| : |t| = r\}}{r} \to 0, \quad \text{as } r \to \infty,$$

then f is constant.

(State any results you use without proof). [Marks

iii. Prove the **Maximum-Modulus Theorem**: Let f be analytic in an open c nected set A. Let γ be a simple closed path that is contained, together with inside, in A. Let

$$M := \sup_{z \in \gamma} |f(z)|.$$

If there exist z_0 inside γ such that $|f(z_0)| = M$, then f is constant through A. Consequently, if f is not constant in A, then

$$|f(z)| < M \quad \text{for all } z \text{ inside } \gamma.$$

[Marks

4. (a) Let $\delta > 0$ and let $f : D^*(z_0;\delta) \to \mathbb{C}$, where $D^*(z_0;\delta) := \{z : 0 < |z - z_0| <$ Define what is meant by f **has a pole of order** m**, at** z_0. [Marks

(b) Prove that if $\text{ord}(f, z_0) = m$ then $f(z) = (z - z_0)^m\, g(z)$ for all $z \in D^*(z_0;\delta)$, some $\delta > 0$, where g is analytic in $D^*(z_0;\delta) := \{z : 0 < |z - z_0| < \delta\}$ and $g(z_0)$

[Marks

(c) Find the residue of the following functions:

i. $\dfrac{z+3}{z^2 - 2z}$;

ii. $\dfrac{z^2}{(z-1)(z-2)(z-3)}$;

iii. $\dfrac{e^z}{z(z-1)^2}$;

iv. $\dfrac{z+1}{z^2(z-3)}$.

[Marks

5. Let f be a analytic in the upper-half plane $\{z : \text{Im}(z) \geqslant 0\}$, except at finitely many points, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leqslant \frac{M}{|z|^\alpha}, \quad |z| \geqslant R \quad \text{with } \text{Im}(z) \geqslant 0.$$

Prove that

$$I := \int_{-\infty}^{\infty} f(x)\, dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of residues of } f \text{ in the upper half plane.}$$

[Marks 40]

Hence evaluate the integral

$$\int_0^\infty \frac{x^2}{(x^2+2)(x^2+3)}\, dx.$$

(You may assume without proof the **Residue Theorem**). [Marks 60]

6. (a) State the **Principle of Argument Theorem**. [Marks 10]

   (b) Prove **Rouche's Theorem**: Let γ be a simple closed path in an open starset A. Suppose that

   i. f, g are analytic in A except for finitely many poles, none lying on γ.

   ii. f and $f+g$ have finitely many zeros in A.

   iii. $|g(z)| < |f(z)|, \quad z \in \gamma$.

   Then

$$ZP(f+g;\gamma) = ZP(f;\gamma),$$

   where $ZP(f+g;\gamma)$ and $ZP(f;\gamma)$ denotes the number of zeros - number of poles inside γ of $f+g$ and f respectively, where each is counted as many times as its order. [Marks 60]

   (c) State the **Fundamental Theorem of Algebra**. [Marks 05]

   (d) Determine the number of zeros of the polynomial $g(z) = 2z^4 - 2z^3 + z^2 - z + 9$ lie in the disc $|z| = 2$. [Marks 25]