EASTERN UNIVERSITY, SRI LANKA

FACULTY OF SCIENCE

Second Year First Semester Examination in Science - 2021/2022

(March/Aprial, 2024)

MT 2022 - Calculus

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wer all Questions | Time: Two hours

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Explain what is meant by the function f has a limit $l(\in \mathbb{R})$ at a point $a(\in \mathbb{R})$. [10 Marks]

Let a function be defined by $f(x) = (x^2 + 2x)$.

Given $\epsilon > 0$ find a $\delta > 0$ such that

$$|f(x) - 15| < \epsilon,$$

for all x with $0 <| x - 3 |< \delta$. Hence prove that $\lim_{x\to 3} f(x) = 15$. [25 Marks]

(b) If $\lim_{x\longrightarrow a} f(x) = l$, then show that $\lim_{x\longrightarrow a} |f(x)| = |l|$. Is the converse true? Justify your answer. [30 Marks]

(c) Let $A \subseteq \mathbb{R}$ and $f : A \longrightarrow \mathbb{R}$ be a function. Prove that $\lim_{x\longrightarrow a} f(x) = l$ if and only if for every sequence (x_n) in A with $x_n \longrightarrow a$ as $n \longrightarrow \infty$ such that $x_n \neq a$ for all $n \in \mathbb{N}$, we have $f(x_n) \longrightarrow l$ as $n \longrightarrow \infty$.

[35 Marks]

(a) Define what it means to say that a real-valued function f is continuous at a point 'a' in its domain.

Using the definition of continuity directly, prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous on $\mathbb{R}$. [35 Marks]

(b) Suppose that $f : A \to \mathbb{R}$ is continuous at c. Prove that if $f(c) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for $x \in (c - \delta,\ c + \delta)$. [20 Marks]

(c) Prove that if a function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, then it is bounded on $[a, b]$. [30 Marks]

(d) State the Intermediate Value Theorem and use it to prove that there exists a number $c \in [0, \frac{\pi}{2}]$ such that $2c - 1 = \sin(c^2 + \frac{\pi}{4})$. [15 Marks]

Q3. (a) Define what it means to say that the real-valued function f is differentiable at a point 'a' in its domain. [10 Marks]

(b) State the Mean-Value Theorem and use it to prove the following:

i. Suppose that $f : [a,\ b] \to \mathbb{R}$ and $g : [a,\ b] \to \mathbb{R}$ are differentiable function on $[a,\ b]$ and that $f'(x) = g'(x)$ for all $x \in [a,\ b]$, then there is some real constant k such that

$$f(x) = g(x) + k \qquad \text{for all } x \in [a,\ b]$$

ii. $x < \sin^{-1} x < \dfrac{x}{\sqrt{1-x^2}}$, $\forall x \in (0, 1)$. [60 Marks]

(c) Suppose that f and g are continuous on $[a, b]$ differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Prove that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

[30 Marks]

Q4. (a) Let f be a real-valued bounded function on [a,b]. Define the following:

i. the Upper integral $\overline{\int_a^b} f$ of f;

ii. the Lower integral $\underline{\int_a^b} f$ of f;

without forgetting to define all the terms you use. [20 Marks]

(b) Let $f : [a,\ b] \longrightarrow \mathbb{R}$ be a bounded function. Suppose that there exist a partition P of $[a,\ b]$ such that

$$U(P, f) - L(P, f) < \epsilon.$$

Show that f is Riemann integrable. [20 Marks]

(c) Prove that if the function $f : [a, b] \to \mathbb{R}$ is bounded and Riemann integrable then $-f$ is also Riemann integrable and

$$\int_a^b f(x)dx = - \int_a^b f(x)dx.$$

[30 Marks]

(d) Prove that the function $f(x) = (2x + 1)$ is Riemann integral on $[-3,\ 0]$ and show that

$$\int_{-3}^{0} (2x + 1)dx = -6.$$

[30 Marks]