EASTERN UNIVERSITY, SRI LANKA

FACULTY OF SCIENCE

SECOND YEAR FIRST SEMESTER EXAMINATION IN SCIENCE - 2021/2022

(Mar./Apr., 2024)

MT2012 - LINEAR ALGEBRA I

Answer all questions | Time: Two hours

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1. (a) i. Define vector addition and scalar multiplication on a set $V = \{x : x \in \mathbb{R}, x > 0\}$ as follows:

$$x \oplus y = xy \text{ and } \alpha \odot x = x^{\alpha} \ \forall x, y \in V, \ \forall \alpha \in \mathbb{R}.$$

Show that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

ii. Let $\mathbb{Z}^3$ be the set of tuples of integers with vector addition '+'and scalar multiplication ' . ' are defined by

$$(l, m, n) + (l', m', n') = (l + l', m + m', n + n'),$$

$$\alpha.(l, m, n) = ([\alpha]\, l, [\alpha]\, m, [\alpha]\, n)$$

where $[\alpha]$ is the integer part of α and $l, m, n, l', m', n' \in \mathbb{Z}$.

Is $(\mathbb{Z}^3, +, .)$ a vector space over the field $\mathbb{R}$? Justify your answer.

(b) Which of the following sets are subspaces of $\mathbb{R}^3$. Justify your answer for each case.

i. $\{(k, l, m) : k^2 = m^2\}$;

ii. $\{(a, b, c) : a^2 + b^2 + c^2 \leq 1\}$.

2. (a) Define the following:

   i. a *linearly independent set* of vectors;

   ii. a *basis* of a vector space;

   (b) Let V be a vector space. Show that

   i. any linearly independent set of vectors of V may be extended to a basis for V;

   ii. if L is a subspace of V, then there exists a subspace M of V such that $V = L \oplus M$, where $\oplus$ denote the direct sum.

   (c) Let V be a vector space over the field $\mathbb{F}$.

   i. let $\{u_1, u_2, \cdots, u_n\}$ be a linearly dependent subset of V with each $u_j \neq 0,\ j = 1, 2, \cdots, n$. Prove that, there exist u_i ($2 \leq i \leq n$) is a linear combination of the preceding vectors.

   ii. let S be a subset of V and $u, v \in V$. If $u \in \langle S \cup \{v\} \rangle$ and $u \notin \langle S \rangle$, then prove that $v \in \langle S \cup \{u\} \rangle$.

3. (a) Define:

   (i) *Range space* $R(T)$;

   (ii) *Null space* $N(T)$

   of a linear transformation T from a vector space V into another vector space W.

   Find $R(T)$, $N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by:

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z)\ \ \forall\ (x, y, z) \in \mathbb{R}^3.$$

   Verify the equation, $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation.

   (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x, y, z) = (x + 2y,\ x + y + z,\ z)$ be a linear transformation and let $B_1 = \{(1, 1, 1),\ (1, 2, 3),\ (2, -1, 1)\}$ and $B_2 = \{(1, 1, 0),\ (0, 1, 1), (1, 0, 1)\}$ be bases of $\mathbb{R}^3$.

   i. find the matrix representation of T with respect to the basis B_1;

ii. using the transition matrix, find the matrix representation of T with respect to the basis B_2.

4. (a) Define the following terms:

   (i) *elementary matrix*;

   (ii) *row reduced echelon form* of a matrix.

   (b) Let A be a non-singular matrix. Prove the following:

   (i) A^{-1} can be obtained by applying the same elementary row operations when transform A into identity matrix;

   (ii) if B is a matrix obtained by performing an elementary row operation on A, then A and B have the same rank.

   (c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

   (d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}.$$