EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2014/2015

FIRST SEMESTER (Sep./Oct., 2016)

103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I (PROPER)

---

ver all questions | Time : Three hours

(a) For any three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, prove that the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c} .$$

Let $\underline{l}$, $\underline{m}$ and $\underline{n}$ be three non zero and non co-planer vectors such that any two of them are not parallel. By considering the vector product $(\underline{r} \wedge \underline{l}) \wedge (\underline{m} \wedge \underline{n})$, prove that any vector $\underline{r}$ can be expressed in the form

$$\underline{r} = (\underline{r} \cdot \underline{\alpha})\underline{l} + (\underline{r} \cdot \underline{\beta})\underline{m} + (\underline{r} \cdot \underline{\gamma})\underline{n}.$$

Hence find the vectors $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ in terms of $\underline{l}$, $\underline{m}$ and $\underline{n}$.

(b) Find the equation of the plane passing through three given terminal points of $\underline{a}$, $\underline{b}$ and $\underline{c}$.

(c) Find the volume of the parallelepiped whose edges are represented by $(2, -3, 4)$, $(1, 2, -1)$ and $(3, -1, 2)$.

2. Define the following terms:

- gradient of a scalar field;
- divergence of a vector field.

(a) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, $r = |\underline{r}|$ and $\underline{a}$ be a constant vector. Find $\text{div}(r^n \underline{r})$, wh
n is a constant. Show that

$$\text{grad}\left(\frac{\underline{a} \cdot \underline{r}}{r^3}\right) = \frac{\underline{a}}{r^3} + 3\frac{(\underline{a} \cdot \underline{r})}{r^5}\,\underline{r}.$$

(b) Find the directional derivative of $\phi = 2x^3 - 3yz$ at the point $(2, 1, 3)$ in
direction parallel to the line whose direction cosines are proportional to $(2, 1,$

(c) Determine the constant 'a' so that the vector

$$\underline{F} = (x + 3y)\underline{i} + (y - 2z)\underline{j} + (x + az)\underline{k}$$

is solenoidal.

3. (a) If $\underline{F} = (2x + y)\underline{i} + (3y - x)\underline{j}$. Evaluate $\int_C \underline{F} \cdot d\underline{r}$ where C is the curve in the plane consisting of the straight line from $(0, 0)$ to $(2, 0)$ and then to $(3, 2)$.

(b) Evaluate $\int\int_S \underline{F} \cdot \underline{n}\, dS$, where $\underline{F} = z\underline{i} + x\underline{j} - 3y^2 z\underline{k}$ and S is the surface of cylinder $x^2 + y^2 = 16$ included in the first octant between the planes $z = 0$
$z = 5$.

(c) State the Divergence theorem, and use it to evaluate $\int\int_S \underline{F} \cdot \underline{n}\, dS$, where $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and S is the surface of the cube bounded by $x = 0$
$x = 1,\ y = 0,\ y = 1,\ z = 0$ and $z = 1$.

4. (a) Prove that the radial and transverse component of the acceleration of a part
in terms of the polar co-ordinates (r, θ) are

$$\ddot{r} - r\dot{\theta}^2 \text{ and } \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \text{ respectively.}$$

(b) A particle of mass m rests on a smooth horizontal table attached through a fixed point on the table by a light elastic string of modules mg and unstretched length 'a'. Initially the string is just taut and the particle is projected along the table in a direction perpendicular to the line of the string with velocity $\sqrt{\frac{4ag}{3}}$. Prove that if r is the distance of the particle from the fixed point at time t then

$$\frac{d^2r}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g(r-a)}{a}.$$

Prove also that the string will extend until its length is $2a$ and that the velocity of the particle is half of its initial velocity.

(a) State the angular momentum principle for motion of a particle.

(b) A right circular cone with a semi vertical angle α is fixed with its axis vertical and vertex downwards. A particle of mass m is held at the point A on the smooth inner surface of the cone at a distance 'a' from the axis of revolution. The particle is projected perpendicular to OA with velocity 'u', where O is the vertex of the cone. Show that the particle rises above the level of A if $u^2 > ag\cot\alpha$ and greatest reaction between the particle and the surface is

$$mg\left(\sin\alpha + \frac{u^2}{ag}\cos\alpha\right).$$

A rocket is fired upwards. Matter is ejected with constant relative velocity gT, at a constant rate $\frac{2M}{T}$. Initially the mass of the rocket is $2M$, half of this is available for ejection. Neglecting air resistance and variation in gravitational attraction, show that the greatest speed of the rocket is attained when the mass of the rocket is reduced to M and this speed is

$$gT\left(\ln 2 - \frac{1}{2}\right).$$

Show also that the rocket will reach the greatest height given by

$$\frac{1}{2}\,gT^2(1-\ln 2)^2.$$