EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST YEAR EXAMINATION IN SCIENCE - 2010/2011

SECOND SEMESTER - (JUNE, 2013)

AM 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

(PROPER & REPEAT)

Answer **All** Questions Time Allowed: **3** Hours

Q1. (a) State the necessary and sufficient condition for the differential equation (DE)

$$M(x,y)\,dx + N(x,y)\,dy = 0\,, \qquad (1)$$

to be *exact*. **[10 Marks]**

(b) Verify that the condition for an exact differential equation (DE) is satisfied by

$$[M(x,y)dx + N(x,y)dy]e^{\int f(x)dx} = 0$$

if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + Nf(x).$$

[20 Marks]

Hence show that an integrating factor can always be found for the DE (1) if

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

is a function of x only. **[10 Marks]**

Hence using this method solve the following DE

$$(y^2 + 3xy + 2y)dx + (xy + x^2 + x)dy = 0.$$

[30 Marks]

(c) If, $\tan x$, is a particular solution of the following nonlinear Riccati equation

$$\frac{dy}{dx} - y^2 - 1 = 0, \qquad (2)$$

then obtain the general solution of the equation (2).

[30 Marks]

Q2. (a) If $F(D) = \sum_{i=0}^{n} p_i D^i$, where $D \equiv \frac{d}{dx}$ and p_i, $i = 1, \ldots, n$, are constants with $p_0 \neq 0$, prove the following formulas:

(i) $\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}$, where α is a constant and $F(\alpha) \neq 0$;

(ii) $\frac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D+\alpha)} V$, where V is a function of x.

[40 Marks]

(b) Find the general solution of the following differential equations by using the results in (a):

(i) $(D^4 - 2D^2 + 1)y = 40 \cosh x$;

(ii) $(D^2 - 6D + 13)y = 8e^{3x} \sin 2x$.

[60 Marks]

Q3. (a) Let $x = e^t$. Show that

$$x\frac{d}{dx} \equiv \mathcal{D}, \quad x^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D},$$

and

$$x^3 \frac{d^3}{dx^3} \equiv \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2),$$

where $\mathcal{D} \equiv \frac{d}{dt}$.

[20 Marks]

Use the above results to find the general solution of the following Cauchy-Euler differential equation

$$(x^3 D^3 + 3x^2 D^2 + xD + 8)y = 65 \cos(\log x),$$

where $D \equiv \frac{d}{dx}$.

[40 Marks]

(b) Define what is meant by *orthogonal trajectories of curves.*

[10 Marks]

Find the orthogonal trajectories of the family of curves

$$r = \frac{a\theta}{1+\theta},$$

where a is a constant.

[30 Marks]

Q4. (a) Define what is meant by the point, '$x = x_0$', being

(i) an *ordinary* ;

(ii) a *singular*;

(iii) a *regular singular*

point of the DE

$$y'' + p(x)y' + q(x)y = 0,$$

where the prime denotes differentiation with respect to x, and $p(x)$ and $q(x)$ are rational functions.

[30 Marks]

(b) (i) Find the regular singular point(s) of the DE

$$4xy'' + 2y' - 7y = 0. \qquad (3)$$

(ii) Use the method of Frobenius to find the general solution of the equation (3).

[70 Marks]

Q5. (a) Solve the following system of DEs:

(i) $\dfrac{dx}{1} = \dfrac{dy}{-2} = \dfrac{dz}{3x^2 \sin(y + 2x)}$;

(ii) $\dfrac{dx}{x^2 + y^2 - yz} = \dfrac{dy}{xz - x^2 - y^2} = \dfrac{dz}{z(x - y)}$.

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz = 0.$$

[5 Marks]

Hence solve the following equation

$$(yz + xyz)\, dx + (xz + xyz)\, dy + (xy + xyz)\, dz = 0. \qquad (4)$$

(You may use the integrating factor $\mu = 1/(xyz)$ for equation (4).)

[15 Marks]

(c) Find the general solution of the following linear first-order partial differential equations:

(i) $x^2p + y^2q = -z^2$;
(ii) $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$.

[30 Marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$2xz - px^2 - 2qxy + pq = 0.$$

Here, $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

[20 Marks]

Q6. (a) Utilize the Fourier series of the function

$$f(x) = |x|, \quad -1 \le x \le 1,$$

to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\pi x = \frac{(2|x| - 1)\pi^2}{4}.$$

[20 Marks]

Hence deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots .$$

[20 Marks]

(b) Use Fourier transform to solve the one-dimensional heat equation

$$\frac{\partial U}{\partial t} - 2\frac{\partial^2 U}{\partial x^2} = 0,$$

subject to the boundary conditions

$$U(0,t) = 0, \; U(x,0) = e^{-x}, \quad x > 0$$

and $U(x,t)$ is bounded where $x > 0$ and $t > 0$.

[40 Marks]

(c) (i) Define the *gamma-function* $\Gamma(x)$ and *beta-function* $B(m,n)$, where m, n are positive integers.

(ii) Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta.$$

(You may use the following results without proof

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.)$$

[20 Marks]

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