EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2011/2012

FIRST SEMESTER (Jan./Feb., 2014)

AM 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

Answer all questions **Time : Three hours**

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1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove that

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}.$$

Hence show that

$$(\underline{a} \wedge \underline{b}) \cdot [(\underline{b} \wedge \underline{c}) \wedge (\underline{c} \wedge \underline{a})] = [\underline{a} \cdot (\underline{b} \wedge \underline{c})]^2.$$

(b) Let the vector $\underline{x}$ be given by the equation $\lambda\underline{x} + \underline{x} \wedge \underline{a} = \underline{b}$, where $\underline{a}, \underline{b}$ are constant vectors and λ is a non-zero scalar. Show that $\underline{x}$ satisfies the equation

$$\lambda^2(\underline{x} \wedge \underline{a}) + (\underline{a} \cdot \underline{b})\underline{a} - \lambda|\underline{a}|^2\underline{x} + \lambda(\underline{a} \wedge \underline{b}) = 0.$$

Hence find $\underline{x}$ in terms of $\underline{a}, \underline{b}$ and λ.

(c) Find the vector $\underline{x}$ and the scalar λ which satisfy the equations

$$\underline{a} \wedge \underline{x} = \underline{b} + \lambda\underline{a}, \quad \underline{a} \cdot \underline{x} = 2,$$

where $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$.

(a) Define the following terms:

i. the gradient of a scalar field ϕ;

ii. the curl of a vector field $\underset{\sim}{A}$.

(b) Prove that if ϕ is a scalar field and $\underset{\sim}{A}$ is a vector field then

$$\text{curl}(\phi \underset{\sim}{A}) = \phi\ \text{curl}\underset{\sim}{A} + \text{grad}\phi \wedge \underset{\sim}{A}.$$

(c) Let $\underset{\sim}{a}$ be a non zero constant vector and let $\underset{\sim}{r}$ be a position vector of a point such that $\underset{\sim}{a} \cdot \underset{\sim}{r} \neq 0$, and let n be a constant. If $\phi = (\underset{\sim}{a} \cdot \underset{\sim}{r})^n$, then show that $\nabla^2 \phi = 0$ if and only if $n = 0$ or $n = 1$.

If $r = |\underset{\sim}{r}|$, find grad $\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{r}}{r^5}\right)$. Hence show that

$$\text{curl}\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{r}}{r^5}\ \underset{\sim}{r}\right) = \frac{\underset{\sim}{a} \wedge \underset{\sim}{r}}{r^5}.$$

(d) i. Find the unit normal vector to the surface $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

ii. Show that $\underset{\sim}{A} = (2xy + z^3)\ \underset{\sim}{i} + x^2\ \underset{\sim}{j} + 3xz^2\ \underset{\sim}{k}$ is a conservative force field.

State the Stokes' theorem.

(a) Verify the Stokes' theorem for a vector $\underset{\sim}{A} = (2x - y)\ \underset{\sim}{i} - yz^2\ \underset{\sim}{j} - \ y^2z\ \underset{\sim}{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

(b) Evaluate $\iiint_V \phi\ dV$, where $\phi = 45x^2y$ and V is the closed region bounded by the planes $4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0.$.

4. A particle A on a smooth table is attached by a string passing through a small hole in the table and carries a particle B of equal mass hanging vertically. The particle A is projected along the table at right angle to the string with velocity $\sqrt{2gh}$ when at a distance $'a'$ from the hole. Here g is the gravitational acceleration and h is a constant. If r is the distance of the particle A from the hole at time t, show the following:

   (a) $\left(\frac{dr}{dt}\right)^2 = gh\left(1 - \frac{a^2}{r^2}\right) + g(a - r)$;

   (b) the particle B will be pulled up to the hole if the total length of the string is less than $\frac{h}{2} + \sqrt{ah + \frac{h^2}{4}}$;

   (c) the tension of the string is $\frac{1}{2}mg\left(1 + \frac{2a^2h}{r^3}\right)$, where m is the mass of each particle.

5. State the angular momentum principle for motion of a particle.

   A right circular cone with a semi vertical angle α is fixed with its axis vertical and vertex downwards. A particle of mass m is held at the point A on the smooth inner surface of the cone at a distance $'a'$ from the axis of revolution. The particle is projected perpendicular to OA with velocity $'u'$, where O is the vertex of the cone. Show that the particle rises above the level of A if $u^2 > ag\cot\alpha$ and greatest reaction between the particle and the surface is

$$mg\left(\sin\alpha + \frac{u^2}{ag}\cos\alpha\right).$$

6. A rocket with initial mass M is fired upwards. Matter is ejected with relative velocity u at a constant rate eM. Let m be the mass of the rocket without fuel. Show that the rocket cannot rise at once unless $eu > g$ and it cannot rise at all unless $eMu > mg$. If it just rises vertically at once, show that its greatest velocity is given by

$$u\ln\left(\frac{M}{m}\right) - \frac{g}{e}\left(1 - \frac{m}{M}\right)$$

   and the greatest height reached is,

$$\frac{u^2}{2g}\left[\ln\left(\frac{M}{m}\right)\right]^2 + \frac{u}{e}\left[1 - \frac{m}{M} - \ln\left(\frac{M}{m}\right)\right].$$