EASTERN UNIVERSITY, SRI LANKA
FIRST EXAMINATION IN SCIENCE 2005/2006
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FIRST SEMESTER
MT 101 - FOUNDATION OF MATHEMATICS

swer all questions | Time:Three hours

(a) Define the terms tautology and contradiction as applied to a logical proposition.

Let p and q be two propositions. Determine whether each of the following is a tautology, a contradiction or neither.

i. $(p \to q) \wedge (\neg\, p \vee q)$.

ii. $(p \to q) \to (p \wedge q)$.

iii. $(p \leftrightarrow q) \leftrightarrow [\neg\,(p \wedge \neg\, q) \wedge \neg(q \wedge \neg\, p)]$.

(b) Test the validity of the following argument:

"If you are a mathematician then you are clever. You are clever and rich. Therefore if you are rich then you are a mathematician."

2. Define the following:

- The difference, $A \setminus B$ of two sets A and B.
- Symmetric difference, $A \triangle B$ of two sets A and B.
- Power set, $P(A)$ of a set A.

(a) Let A, B and C be three subsets of a universal set X.

(1) Prove that $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

(2) Prove that $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$.

(3) $(A \Delta B) \cap (A \cap B) = \phi$.

(4) Construct a suitable example to show that $A \cup (B \setminus C) \neq (A \cup B) \setminus (A \cup C)$.

Q3. (a) What is meant by an equivalence relation on a set?

Let A be any set and let R be an equivalence relation on A. Prove the following:

(i) $[a] \neq \phi \quad \forall a \in A$.

(ii) $aRb \iff [a] = [b]$.

(iii) $b \in [b] \iff [a] = [b]$.

(iv) For any $a, b \in A$ either $[a] = [b]$ or $[a] \cap [b] = \phi$.

(Here $[x]$ denotes the equivalence class of x.)

(b) Define a relation R on $\mathbb{N} \times \mathbb{N}$ by $(a, b)R(c, d)$ if and only if $a + b = c + d$.

(a) Prove that R is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

(b) Let S denote the set of equivalence classes of R. Show that there is a one-to-one and onto function from S to $\mathbb{N}$.

Q4. Define the terms 'Injective' and 'Surjective' as applied to a mapping.

(a) Let $f_1 : A \to B$ and $f_2 : B \to A$ be mappings such that $f_2 \circ f_1 = I_A$ and $f_1 \circ f_2 = I_B$, where I_A and I_B are the identity mappings defined on A and B respectively. Prove that f_1 is bijective and $f_2 = f_1^{-1}$.

(b) Let $f : S \to T$ be a mapping. Prove that f is injective if and only if $f(A) \cap f(S \setminus A) = \phi, \quad \forall A \subseteq S$.

(c) Give an example of a function f from $\mathbb{N}$ to $\mathbb{N}$ such that:

(1) f is injective but not surjective;

(2) f is surjective but not injective.

Q5. (a) Define the following terms:

(i) Partially ordered set;

(ii) Totally ordered set.

(b) Let R be a relation defined on $\mathbb{N}$ by xRy if and only if x divides y.

(i) Show that R is a partial order relation on $\mathbb{N}$.

(ii) Find the infimum and supremum (if exists) for a subset $A = \{2, 4, 8, 12\}$ of $\mathbb{N}$.

(c) (i) What is meant by a countable set?

(ii) Prove that $\mathbb{N} \times \mathbb{N}$ is countable.

6. (a) Define the term, greatest common divisor (gcd) of two integers.

Let $\gcd(a, b) = d$, where a, b are two integers not both zero. Prove that $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$.

(b) With the usual notations, prove that the linear Diophantine equation $ax+by = c$ has a solution if and only if $\gcd(a, b)$ divides c.

Further show that, if $\gcd(a, b)$ divides c then it has infinitely many solutions of the form $x = \frac{b}{\gcd(a,b)}k + x_0$ and $y = -\frac{a}{\gcd(a,b)}k + y_0$, where x_0, y_0 is a particular solution and $k \in \mathbb{Z}$.

(c) Solve the congruence

$2x + 11 = 7(\text{mod}3)$