EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 2010/2011

FIRST SEMESTER (April, 2013)

PM 203 - EIGENSPACE AND QUADRATIC FORMS (PROPER & REPEAT)

Answer all Questions **Time: Two hours**

1. Define the terms eigenvalue and eigenvector of a linear transformation.

   (a) i. Prove that eigenvectors that correspond to distinct eigenvalues of a linear transformation $T: V \to V$ are linearly independent.

   ii. Let λ be an eigenvalue of an operator $T: V \to V$. Let V_λ denotes the set of all eigenvectors of T belonging to the eigenvalue λ. Show that V_λ is a subspace of V.

   (b) Find all eigenvalues and a basis of each eigenspace of the operator $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y, y - z, 2y + 4z)$.

2. Define the term minimum polynomial of a square matrix.

   (a) State the Cayley - Hamilton theorem.

   Find the minimum polynomial of the square matrix

$$\begin{pmatrix} 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

(b) Prove that for any square matrix A, the minimum polynomial exists and is unique.

(c) Let $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A and B are square matrices. Show that the minimum polynomial $m(t)$ of M is the least common multiple of the minimum polynomials $g(t)$ and $h(t)$ of A and B, respectively.

3. (a) Find an orthogonal transformation which reduces the following quadratic form to a diagonal form

$$2x_1^2 - 2x_1x_3 + 2x_2^2 - 2x_2x_3 + 3x_3^2 = 16.$$

(b) Simultaneously diagonalize the following pair of quadratic forms

$$\phi_1 = x_1^2 + 2x_2^2 + 8x_2x_3 + 12x_1x_2 + 12x_1x_3,$$

$$\phi_2 = 3x_1^2 + 2x_2^2 + 5x_3^2 + 2x_2x_3 - 2x_1x_3.$$

4. (a) What is meant by an inner product on a vector space.

Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, where $x_i, y_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Let the inner product $< ., . >$ be defined on $\mathbb{R}^n$ as

$$< x, y >= xy^T = \sum_{i=1}^{n} x_i y_i.$$

Show that $(\mathbb{R}^n, < ., . >)$ is an inner product space.

(b) State and prove Cauchy - Schwarz Inequality.

(c) State the Gram Schmidt Process.

Find the orthonormal set for span of M in $\mathbb{R}^4$, where

$$M = \{(1, 0, -1, 0)^T, (0, 1, 2, 1)^T, (2, 1, -1, 0)^T\}.$$