EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
FIRST EXAMINATION IN SCIENCE -2009/2010
SECOND SEMESTER (April/May, 2012)
PM 102 - REAL ANALYSIS

Answer all Questions

Time: Three hours

1. (a) Define the terms **Supremum** and **Infimum** of a bounded subset A of $\mathbb{R}$.

[10marks]

(b) Prove that an upper bound u of a non-empty set S in $\mathbb{R}$ is the supremum of S if, and only if, for each $\epsilon > 0$ there exists $x_0 \in S$ such that $u - \epsilon < x_0$.

[30marks]

(c) State the Archimedian principle and use it to prove that there exists a positive real number x such that $x^2 = 2$.

[40marks]

(d) Use the Mathematical induction principle to show that
$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

[20marks]

2. (a) Define what is meant by the following terms applied to a sequence of real numbers:

i. bounded;

ii. convergent;

iii. monotone.

[15marks]

(b) Prove that every increasing sequence of real numbers which is bounded above is convergent. [35marks]

(c) Let (y_n) be a sequence of real numbers defined inductively by

$$y_1 = 1, \qquad y_{n+1} = \frac{1}{4}(2y_n + 3) \text{ for all } n \in \mathbb{N}.$$

Show that (y_n) is convergent and $\lim_{n\to\infty} y_n = \frac{3}{2}$. [50marks]

3. (a) i. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Explain what is meant by the function f has a limit $l(\in \mathbb{R})$ at a point $a(\in \mathbb{R})$. [15marks]

ii. Use the definition of the limit to show that $\lim_{x\to -1} \frac{x+5}{2x+3} = 4$. [25marks]

(b) i. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Let $a \in \mathbb{R}$. Prove that $\lim_{x\to a} f(x) = l$ exists finitely if, and only if, for every sequence (x_n) in A that converges to a such that $x_n \neq a$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to l. [40marks]

ii. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = \sin(1/x)$ $\forall\, x \neq 0$. Show that $\lim_{x\to 0} f(x)$ does not exists in $\mathbb{R}$. [20marks]

4. (a) i. Define what is meant by the statement that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point a $(\in \mathbb{R})$. [15marks]

ii. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x$, $\forall x \in \mathbb{R}$ is continuous at every point in $\mathbb{R}$. [25marks]

(b) Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Prove that if $f : I \to \mathbb{R}$ is continuous on I then f is bounded on I. [40marks]

(c) State the Intermediate Value Theorem and use it to prove that the equation $2x^2(x+2) - 1 = 0$ has a root in each of the intervals $(-2, -1)$, $(-1, 0)$ and (0,1). [20marks]

5. (a) i. Define what is meant by a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at the point $x_0 \in \mathbb{R}$. [15marks]

ii. Discuss differentiability of each of the following functions $f : \mathbb{R} \to \mathbb{R}$ at the origin:

1. $f(x) = \sin\ x$
2. $f(x) = |x|$
3. $f(x) = \begin{cases} 3 + x, & x \leq 0; \\ 3 - x, & x > 0. \end{cases}$ [30marks]

(b) i. Let $f : [a, b] \to \mathbb{R}$ be a function where $a, b \in \mathbb{R}$ with $a < b$. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(You may use the Rolle's Theorem without proving it.) [30marks]

ii. Show that $x < \sin^{-1} x < \dfrac{x}{\sqrt{1 - x^2}} \quad \forall x \in (0, 1)$. [25marks]

6. (a) Suppose that f and g are two continuous real valued functions defined on $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. Suppose also that f and g are differentiable on (a, b) and $g'(x) \neq 0 \quad \forall x \in (a, b)$. Prove that for some $c \in (a, b)$,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

(You may use the Rolle's Theorem without proving it.) [30marks]

(b) i. Suppose that f and g are continuous on $[a, b]$, differentiable on (a, b) and let $f(c) = g(c) = 0$ for some $c \in (a, b)$. Further suppose that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b) \setminus \{c\}$. If $\lim_{x \to c} \dfrac{f'(x)}{g'(x)} = l$ exists finitely prove that $\lim_{x \to c} \dfrac{f(x)}{g(x)} = l$. [20marks]

ii. Prove that $\lim_{x \to 0} \dfrac{(1 - \cos\ x)}{x^2} = \dfrac{1}{2}$. [15marks]

(c) State the Taylor's Theorem and use it to prove that

$$1 - \frac{1}{2}x^2 \leq \cos\ x \quad \forall x \in \mathbb{R}.$$

[35marks]