EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN SCIENCE - 2004/2005

Advanced Quantum Mechanics - PH 402

Answer ALL questions.

Time: 3 hours

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1. (a) If the momentum and position operators in one-dimension are given by

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}, \quad \text{and} \quad \hat{x} = x,$$

find the value of the commutators $[\hat{x}, \hat{p}]$, $[\hat{x}, \hat{p}^2]$ and $[\hat{x}^2, \hat{p}]$.

Explain why it is not possible to measure simultaneously the energy and the position of a free particle.

(b) The wave function of a particle executing linear harmonic motion in the lowest eigenstate is given by

$$\psi(x) = \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha^2 x^2}{2}}.$$

Show that the uncertainties in position and momentum of the particle are

$$\Delta x = \frac{1}{\sqrt{2}\,\alpha}, \qquad \Delta p = \frac{\alpha\hbar}{\sqrt{2}},$$

such that

$$\Delta x \Delta p = \frac{\hbar}{2}.$$

The values of the following integrals are given:

$$\int_{-\infty}^{\infty} e^{-\alpha^2 x^2}\, dx = \sqrt{\frac{\pi}{\alpha^2}}\ ; \qquad \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2}\, dx = \frac{1}{2\alpha^2}\sqrt{\frac{\pi}{\alpha^2}}\ .$$

2. Particles of mass m are moving with velocity v along the positive direction in the x-axis and encounter a potential barrier defined by

$$\begin{aligned} V(x) &= 0 & x < 0 \\ V(x) &= V_0 & 0 < x < a \\ V(x) &= 0 & x > a \end{aligned}$$

(i) Determine the transmission coefficient T for the cases (a)$V_0 > \frac{1}{2}mv^2$ and (b) $V_0 < \frac{1}{2}mv^2$

(ii) Show that there is a discrete set of energies for which the barrier is completely transparent and obtain an expression for these energies.

(iii) Show that the minimum velocity v_{min} for which the barrier will be completely transparent as

$$v_{min} = \sqrt{\frac{2}{m}\left\{V_0 + \frac{h^2}{8ma^2}\right\}}$$

3. If $\hat{L}_x, \hat{L}_y, \hat{L}_z$ are the components of the angular momentum vector operator $\underline{\hat{L}}$, show that $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$, $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$, and $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$. Hence show that $[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0$. What can you infer form the above for the measurement of angular momentum.

If we define the ladder operates $\hat{L}_+$ and $\hat{L}_-$ as $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ then show that

(i) $\hat{L}_\pm\hat{L}_\mp = \hat{L}^2 - \hat{L}_z^2 \pm \hbar\hat{L}_z$

(ii) $[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm$

(iii) $[\hat{L}^2, \hat{L}_\pm] = 0$

Using the above relations show that if b is an eigen value of $\hat{L}_z$ corresponding to an eigen function ψ, then $(b \pm \hbar)$ is also an eigen value of $\hat{L}_z$ corresponding to the eigen function $\hat{L}_\pm\psi$.

4. Write down the matrix representation of L_x, L_y and L^2 for $l = 1$ using the relations

$$\langle l, m'|\hat{L}_z|l, m\rangle = \hbar m \delta_{m',m}$$
$$\langle l, m'|\hat{L}_\pm|l, m\rangle = \hbar\{l(l+1) - m(m \pm 1)\}^{\frac{1}{2}}\delta_{m',m\pm1}$$

obtain the eigenfunctions and eigenvalues of L_x.
A particle in p-state is represented by the state vector

$$\Psi = \frac{1}{5}\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

Calculate the probability that a measurement of L_x will yield the value $\hbar$. What are the expectation values of L_x, L_y and L^2?

5. Write down the shrödinger equation for a hydrogen like atom in a state of zero orbital angular momentum. The normalized wave function of the ground state of a hydrogen like atom with nuclear charge Ze, has the form

$$u(r) = A exp(-\beta r)$$

where A and β are constant and r is the distance between the electron and the nucleus. Show the following for the ground state of the atom.

(a) $A^2 = \dfrac{\beta^3}{\pi}$

(b) $\beta = \dfrac{Z}{a_0}$, where $a_0 = \dfrac{4\pi\epsilon_0\hbar^2}{m_e e^2}$

(c) the energy is $E = -Z^2 E_0$, where $E_0 = \dfrac{m_e}{2\hbar^2}\left[\dfrac{e^2}{4\pi\epsilon_0}\right]^2$

(d) the expectation value of r is $\dfrac{3a_0}{2Z}$

The following information may be useful

(i) $\displaystyle\int_0^\infty r^n e^{-\frac{r}{a}} dr = n! a^{n+1}$ where n is a positive integer

(ii) $\nabla^2 = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\left(\dfrac{\partial^2}{\partial\phi^2}\right)$

6. Outline the perturbation theory applicable for the stationary state problems and deduce expressions for the first order perturbation corrections when the energy levels of the unperturbed state are non-degenerate.

Consider a particle of mass m bound by the simple harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$ is in its ground state. A weak perturbation $V_1(x) = Cx^4$ is now adds to this potential. Calculate the first order shift for the ground state wave function for this simple harmonic oscillator given by

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

You may assume that

$$\int_{-\infty}^{\infty} x^4 exp(-\alpha x^2)dx = \frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$$