EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE EXAMINATION IN SCIENCE - ~~2008/20~~ 2010/2

SECOND YEAR FIRST SEMESTER(March/May, 2017)

EXTMT 203 - EIGENSPACES AND QUADRATIC FORMS (REPEAT)

Answer all Questions

Time: Two hours

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1. (a) Define what is meant by the terms *eigenvalue* and *eigenvector* of a linear transformation $T : V \to V$, where V is a vector space.

   Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 3 \end{pmatrix}.$$

   (b) i. Prove that the eigenvectors that corresponding to distinct eigenvalues of a linear transformation are linearly independent.

   ii. Prove that an $n \times n$ matrix A is similar to diagonal matrix D if and only if A has n linearly independent eigenvectors, where the diagonal entries of D are the corresponding eigenvalues of A.

   iii. Let A and B be n-square matrices. Show that AB and BA have the same eigenvalues.

   (c) Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

2. Define what is meant by the term *minimum polynomial* of a square matrix.

(a) State and prove the *Cayley-Hamilton* theorem.

Find the minimum polynomial of the square matrix

$$\begin{pmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

(b) Prove that if $m(t)$ is a minimum polynomial of an $n \times n$ matrix A and $\psi_A(t)$ is the characteristic polynomial of A, then $\psi_A(t)$ divides $[m(t)]^n$.

(c) Prove that for any square matrix A, the minimum polynomial exists and is unique.

3. (a) Find an orthogonal transformation which reduces the following quadratic form to a diagonal form

$$5x_1^2 + 6x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3 = 1.$$

(b) Simultaneously diagonalize the following pair of quadratic forms

$$\phi_1 = x_1^2 - x_2^2 - 2x_3^2 - 2x_1x_2 + 4x_2x_3,$$

$$\phi_2 = x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3.$$

4. (a) Define what is meant by an *inner product* on a vector space.

Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, where $x_i, y_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Let the inner product $< ., . >$ be defined on $\mathbb{R}^n$ as

$$< x, y >= xy^T = \sum_{i=1}^{n} x_i y_i.$$

Show that $(\mathbb{R}^n, < ., . >)$ is an inner product space.

(b) Prove that if non zero vectors $\{x_1, x_2, ..., x_n\}$ in an inner product space V are mutually orthogonal, then they are linearly independent.

(c) State the *Gram Schmidt* process.

Find the orthonormal set for span of M in $\mathbb{R}^4$, where

$$M = \{(1, 0, -1, 0)^T, (0, 1, 2, 1)^T, (2, 1, -1, 0)^T\}.$$