EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE -2009/2010

SECOND SEMESTER (April /May, 2012)

MT 202 – METRIC SPACE

Answer all questions **Time allowed: 02 Hours**

1. Define the terms **metric** space and **complete metric space.** [15 Marks]

a. Let l^∞ be the set of all bounded sequences of complex numbers; that is,
$l^\infty = \{ x = (x_i)_{i=1}^{\infty} \mid x_i \in \mathbb{C} \text{ and } \exists\, c_x \in \mathbb{R} \text{ such that } |x_i| < c_x \ \forall\, i \in \mathbb{N}\}$.
Define $d: l^\infty \times l^\infty \to [0, \infty)$ by
$d(x, y) = \operatorname{Sup}_{i \in \mathbb{N}} |x_i - y_i|$, where $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty} \in l^\infty$.

Prove that (l^∞, d) is a complete metric space. [40 Marks]

b. Let $C_{[0,1]}$ be the set of all continuous real valued functions on [0, 1]; that is,
$C_{[0,1]} = \{ f: [0,1] \to \mathbb{R} \mid f \text{ is continuous on } [0,1] \}$.
Define $d: C_{[0,1]} \times C_{[0,1]} \to [0, \infty)$ by
$d(f, g) = \int_0^1 |f(t) - g(t)| dt$, for $f, g \in C_{[0,1]}$.
Prove that $(C_{[0,1]}, d)$ is **not** a complete metric space. [45 Marks]

2.

a. Let (X, d) be a metric space and $a \in X$.

I. Show that the open ball $B(a, r) = \{x \in X \mid d(a, x) < r\}$ is an open set for any real number $r > 0$. [20 Marks]

II. Show that singleton sets are closed in any metric space. [15 Marks]

Is it true that singleton sets are not open in any metric space? Justify your answer. [10 Marks]

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b. Let (X, d) be a metric space and let A and B be two subsets of X.

I. Define terms **interior of** A (A^0) and **closure of** A ($\overline{A}$). [10 Marks]

II. Prove the following:

- $(A \cap B)^0 = A^0 \cap B^0$;
- $\overline{(X \setminus A)} = X \setminus A^0$;
- $X \setminus \bar{A} = (X \setminus A)^0$. [15 x 3 Marks]

3. What is meant by a function f from a metric space (X, d_1) to a metric space (Y, d_2) is continuous at a point $a \in X$? [10 Marks]

a. Prove that f is continuous on X if, and only if, whenever G is an open set in Y, $f^{-1}(G)$ is open in X. [30 Marks]

Is it true that, if f is continuous on X then the image $f(A)$ of every open set A in X is open in Y? Justify your answer. [15 Marks]

b. Prove that the following conditions are equivalent:

- f is continuous on X;
- $f^{-1}(F)$ is closed in X whenever F is closed in Y;
- $f^{-1}(B^0) \subseteq \left(f^{-1}(B)\right)^0$ for every $B \subseteq X$.

[15 x 3 Marks]

4. Let (X, d) be a metric space and let $f: X \rightarrow X$ be a function.

a. Let A be a compact subset of X and let $a \in X \setminus A$. Prove that there exist open sets G and H in X such that $a \in G$, $A \subseteq H$ and $G \cap H = \varphi$. [30 Marks]

b. Prove that, if A is a compact subset of X then A is closed and bounded. [30 Marks]

c. Is it true that, if A is a closed and bounded subset of X then A is compact? Justify your answer. [15 Marks]

d. Let A be a compact subset of X and let f be continuous on X. Prove that $f(A)$ is a compact subset of X.

[25 Marks]