EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2016/2017)

FIRST SEMESTER (Nov./Dec., 2018)

PM 201 - VECTOR SPACES AND MATRICES

Answer all questions | Time: Three hours

1. Define the term *vector space*.

   (a) Let $V = \{f : \ f : \mathbb{R} \to \mathbb{R}, \ f(x) > 0, \ \forall \ x \in \mathbb{R}\}$. For any $f, f_1, f_2 \in V$ and for any $\alpha \in \mathbb{R}$ define an addition $\oplus$ and a scalar multiplication $\odot$ on V as follows:

   $$(f_1 \oplus f_2)(x) = f_1(x).f_2(x), \ \forall x \in \mathbb{R}$$

   and

   $$(\alpha \odot f)(x) = (f(x))^{\alpha}.$$

   Prove that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

   (b) Let $S_1, \ S_2$ be two subspaces of a vector space V over a field F. Prove that $S_1 + S_2$ is the smallest subspace of V containing both S_1 and S_2.

   (c) Determine which of the following sets are subspaces of $\mathbb{R}^3$:

   i. $\{(a, b, c) \in \mathbb{R}^3 : a^2 = c^2\}$;

   ii. $\{(0, \alpha, \alpha + 1) : \alpha \in \mathbb{R}\}$.

2. (a) Let V be an $n-$dimensional vector space. Prove the following:

   (i) A linearly independent set of vectors of V with n elements is a basis for V;

   (ii) Any linearly independent set of vectors of V can be extended as a basis for V.

(b) Let V be a vector space over a field F.

i. If $v_1, v_2, \cdots, v_m$ are linearly dependent vectors and $v_1, v_2, \cdots, v_{m-1}$ are linearly independent vectors, then prove that $v_m \in \langle\{v_1, v_2, \cdots, v_{m-1}\}\rangle$.

ii. If $\{u, v\}$ is a basis for a subspace S of V, show that $\{u + 2v, -3v\}$ is also a basis for S.

iii. Let $u_1, u_2, \cdots, u_r$ be linearly independent vectors in V and let $u(\neq 0) \in V$. Prove that $u_i \in \langle\{u, u_1, u_2, \cdots, u_{i-1}\}\rangle$ for some integer i, where $1 \leq i \leq r$ if and only if $u \in \langle\{u_1, u_2, \cdots, u_r\}\rangle$.

3. (a) State the *dimension theorem* for two subspaces of a finite-dimensional vector space. Let U_1 and U_2 be two subspaces of a vector space V. If dim $U_1 = 3$, dim $U_2 = 4$, dim $V = 6$, show that $U_1 \cap U_2$ contains a non-zero vector.

If dim $U_1 = 2$, dim $U_2 = 4$, dim $V = 6$, show that $U_1 + U_2 = V$ if and only if $U_1 \cap U_2 = \{0\}$.

(b) Let V be a vector space over a field F.

i. If L is a subspace of V, prove that there exists a subspace M of V such that $V = L \oplus M$, where $\oplus$ denotes the direct sum.

ii. Let $T : V \to F$ be a linear transformation and let $v \notin N(T)$, where $N(T)$ is the null space of T. Prove that

$$V = \text{span}\{v\} \oplus N(T).$$

4. (a) Define the *range space*, $R(T)$ and the *Null space*, $N(T)$ of a linear transformation T from a vector space V into another vector space W.

(b) Find $R(T)$ and $N(T)$ of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by:

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z) \;\; \forall \; (x, y, z) \in \mathbb{R}^3.$$

Verify the equation, $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation T, where $V = \mathbb{R}^3$.

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by:

$T(x, y, z) = (x + 2y,\ x + y + z,\ z)$ and let $B_1 = \{(1, 1, 1), (1, 2, 3), (2, -1, 1)\}$ and $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ be bases for $\mathbb{R}^3$. Find the following:

(i) The matrix representation of T with respect to the basis B_1;

(ii) The matrix representation of T with respect to the basis B_2 by using the transition matrix.

5. (a) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

(b) Find the determinant of the matrix $\begin{pmatrix} -3a & -3b & -3c \\ d & e & f \\ g - 4d & h - 4e & i - 4f \end{pmatrix}$, if the determinant of the matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is equal to -6.

(c) Find the rank of the matrix for each possible value of the scalar α

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & \alpha + 1 & 3 & \alpha - 1 \\ -3 & \alpha - 2 & \alpha - 5 & \alpha + 1 \\ \alpha + 2 & 2 & \alpha + 4 & -2\alpha \end{pmatrix}.$$

6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Reduce the augmented matrix of the following system of linear equations to its row reduced echelon form and hence determine the conditions on non-zero scalars $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 such that the system has

(i) a unique solution;

(ii) no solution;

(iii) more than one solution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

(b) Show that the system of equations

$$\begin{aligned} x_1 - 3x_2 + x_3 + \alpha x_4 &= a \\ x_1 - 2x_2 + (\alpha - 1)x_3 - x_4 &= 2 \\ 2x_1 - 5x_2 + (2 - \alpha)x_3 + (\alpha - 1)x_4 &= 3a + 4 \end{aligned}$$

is consistent, for all values of a when $\alpha \neq 1$. Find the value of a for which the system is consistent when $\alpha = 1$ and obtain the general solution for these values.

(c) Prove *Crammer's* rule for 3×3 matrix and use it to solve the following system of linear equations:

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 5 \\ 3x_1 + 2x_2 - 3x_3 &= 13 \\ 2x_1 - 5x_2 + x_3 &= 2. \end{aligned}$$