EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2015/2016

FIRST SEMESTER (MARCH/APRIL, 2019)

AM 306 - PROBABILITY THEORY

Answer all questions

Time: Two hours

Statistical table will be provided

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1. (a) State Bayes' theorem and prove it. [30 marks]

   (b) Random variable X is having an Exponential distribution with parameter λ. Then, prove the followings:

   i. expectation of X, $E(X) = \frac{1}{\lambda}$;

   ii. variance of X, $V(X) = \frac{1}{\lambda^2}$;

   iii. moment generating function of X, $M_X(t) = \frac{\lambda}{\lambda - t}$. [40 marks]

   (c) Random variable X is having a Normal distribution with mean 400 and variance 100. Find the following probabilities:

   i. P(X< 370);

   ii. P(X> 380);

   iii. P(370 $\leq$ X $\leq$ 390). [30 marks]

2. (a) The probability density function of the random variable X is given by

$$f_X(x) = \begin{cases} C\,x^2, & \text{if } 0 \leq x \leq 3 \\ 0 \quad , & \text{otherwise} \end{cases}$$

   where C is a constant.

[P.T.O.]

[Question 2(a) continued....]

Find the followings:

i. value of C;

ii. variance of X, $V(X)$;

iii. cumulative probability distribution function of X, $F_X(x)$;

iv. P(X>2.5). [40 ma

(b) The joint probability density function of X and Y is given by,

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{3}(x+2y), & 0 \le x \le 1, 0 \le x \le 1 \\ 0 & , \text{otherwise.} \end{cases}$$

Find the followings:

i. P(-2 $\le x \le$ 0.5, 0.15 $\le y \le$ 0.75);

ii. marginal probability density function of Y, $f_Y(y)$;

iii. conditional probability density function of X given Y, $f_{X/Y}(x,y)$;

iv. conditional expectation of X given Y, $E(X/Y)$. [60 marl

3. (a) Let $x_1, x_2, ..., x_n$ be a set of observations for the random variables $X_1, X_2, ..., X$ each having the following probability density function:

$$f_X(x,\theta) = \begin{cases} \lambda \exp(-\lambda x), & \text{if } x > 0 \\ 0 & , \text{otherwise.} \end{cases}$$

Find the probability density function of

$$Y = max(x_1, x_2, ..., x_n),$$

by using its cumulative probability distribution function. [30 mark

(b) Let $x_1, x_2, ..., x_n$ be a random sample from Normal distribution with mean μ ar variance σ^2. By using its moment generating function, find the probability densi function of the sample mean defined as

$$\overline{X} = \frac{\sum x_i}{n}.$$ [30 mark

(c) Use the method of transformation(change of variables) to find the density function of Z defined as,

$$Z = Y - X;$$

where X and Y are two random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 3x, & \text{if } 0 \leq y \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

[60 marks]

4. (a) Let $x_1, x_2, ..., x_n$ be a random sample of size n from a Normal distribution with unknown mean μ and variance σ^2. Find the moment estimators of μ and σ^2. Estimate σ^2 with the sample data: [7.1, 2.6, 3.0, 9.0, 10.5, 6.2, 5.0, 4.25, 7.7,2.2].

[30 marks]

(b) Let $x_1, x_2, ..., x_n$ be a random sample from a distribution with probability density function,

$$f(\underset{\sim}{x}, \theta) = \begin{cases} 3\theta x^2 \exp(-\theta x^3), & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the Maximum likelihood estimator of θ. [40 marks]

(c) Let $x_1, x_2, ..., x_n$ be a random sample from a Normal distribution with unknown mean μ and unknown variance σ^2. Derive a (1-α)100% confidence interval for population mean μ. [30 marks]

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