EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2016/2017

FIRST SEMESTER (March/April, 2019)

PM 302 - COMPLEX ANALYSIS

Answer all questions

Time: Three hours

1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$.

(b) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ-neighborhood of a point $z_0 = x_0 + iy_0$. Suppose that the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood and that they are continuous at (x_0, y_0). Prove that, if those partial derivatives satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at $z_0 = x_0 + iy_0$, then the derivative $f'(z_0)$ exists.

(c) i. Define what is meant by the function, $h : \mathbb{R}^2 \to \mathbb{R}$, being **harmonic**.

ii. Obtain a harmonic conjugate $v(x, y)$ of a harmonic function $u(x, y) = \dfrac{x}{x^2 + y^2}$ such that $f(x) = u(x, y) + iv(x, y)$ is analytic.

2. (a) Let $D(a;r) := \{z \in \mathbb{C} : |z-a| < r\}$ denotes the open disc with center $a \in \mathbb{C}$ and radius $r > 0$ and let f be analytic on $D(a;r)$ and $0 < s < r$. Prove **Cauchy's Integral Formula**,

$$f(z_0) = \frac{1}{2\pi i} \int_{C(a;s)} \frac{f(z)}{z - z_0}\, dz, \qquad \text{for } z_0 \in D(a;s),$$

where $C(a;s)$ denotes the circle with center a and radius $s > 0$.

(b) By using the **Cauchy's Integral Formula** compute the following integrals:

i. $\displaystyle\int_{C(0;2)} \frac{\sin z}{z+1}\, dz;$

ii. $\displaystyle\int_{C(0;2)} \frac{z\, e^z}{(z^2+i)}\, dz.$

3. (a) State the **Mean-Value Property for Analytic Function**.

(b) i. Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being **entire**.

ii. Prove **Liouville's Theorem**: If f is entire and

$$\frac{\max\{|f(t)| : |t| = r\}}{r} \to 0, \text{ as } r \to \infty,$$

then f is constant.

(State any results you use without proof).

iii. Prove the **Maximum-Modulus Theorem**: Let f be analytic in an open connected set A. Let γ be a simple closed path that is contained, together with its inside, in A. Let

$$M := \sup_{z \in \gamma} |f(z)|.$$

If there exist z_0 inside γ such that $|f(z_0)| = M$, then f is constant throughout A. Consequently, if f is not constant in A, then

$$|f(z)| < M \quad \text{for all } z \text{ inside } \gamma.$$

d a) Let $\delta > 0$ and let $f : D^*(z_0;\delta) \to \mathbb{C}$, where $D^*(z_0;\delta) := \{z : 0 < |z - z_0| < \delta\}$.

s Define what is meant by f has a pole of order m at z_0.

b) Prove that if $\mathrm{ord}(f, z_0) = m$ then $f(z) = (z - z_0)^m\, g(z)$ for all $z \in D^*(z_0;\delta)$, for some $\delta > 0$, where g is analytic in $D^*(z_0;\delta) := \{z : 0 < |z - z_0| < \delta\}$ and $g(z_0) \neq 0$.

(c) Find the value of the integral

$$\int_C \frac{z^2+1}{(z-2)(z^2+4)},$$

where C is taken counterclockwise around the circle,

i. $|z - 3| = 2$;

ii. $|z| = 3$.

Let f be a analytic in the upper-half plane $\{z : \mathrm{Im}(z) \geqslant 0\}$, except at finitely many points, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leqslant \frac{M}{|z|^\alpha}, \quad |z| \geqslant R \quad \text{with } \mathrm{Im}(z) \geqslant 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x)\, dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of } f \text{ in the upper half plane.}$$

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\, dx.$$

(You may assume without proof the **Residue Theorem**).

6. (a) State the **Principle of Argument Theorem.**

(b) Prove **Rouche's Theorem:** Let γ be a simple closed path in an open starset

Suppose that

i. f, g are analytic in A except for finitely many poles, none lying on γ.

ii. f and $f + g$ have finitely many zeros in A.

iii. $|g(z)| < |f(z)|, \quad z \in \gamma$. Then

$$ZP(f + g; \gamma) = ZP(f; \gamma)$$

where $ZP(f + g; \gamma)$ and $ZP(f; \gamma)$ denote the number of zeros - number of pol inside γ of $f + g$ and f respectively, where each is counted as many times as i order.

(c) State the **Fundamental Theorem of Algebra.**

(d) Determine the number of zeros of $p(z) = z^4 - 2z^3 + 9z^2 + z - 1$ in the open unit di $D(0; 2)$.