# EASTERN UNIVERSITY, SRI LANKA
# DEPARTMENT OF MATHEMATICS
# THIRD EXAMINATION IN SCIENCE - 2011/2012
# SECOND SEMESTER (March, 2014)
# MT 301 - GROUP THEORY
# (SPECIAL REPEAT)

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**Answer all Questions** **Time: Three hours**

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1. (a) Define the following terms:

   i. a group;

   ii. a subgroup of a group;

   iii. cyclic group.

   (b) Let $H$ be a non - empty subset of a group $G$. Prove that, $H$ is a subgroup of $G$ if and only if, $ab^{-1} \in H, \forall a, b \in H$.

   (c) Let $H$ be any subgroup of a group $G$ and $a, b \in G$. Prove that $Ha = Hb$ if and only if $ab^{-1} \in H$.

   (d) Let $H$ and $K$ be subgroups of a group $G$. Prove that $HK$ is a subgroup of $G$ if and only if $HK = KH$.

2. (a) State and prove the Lagrange's theorem for a finite group.

   (b) Let $G$ be a group and let $H$ and $K$ be two subgroups of $G$ such that $\mid H \mid = 12$ and $\mid K \mid = 5$. Prove that $H \cap K = \{e\}$.

   (c) Let $G$ be a finite group and $a \in G$. Prove that $O(a)/O(G)$.

   (d) Let $G$ be a finite group of order $n$. Prove that $a^n = e, \forall a \in G$.

3. (a) State the first isomorphism theorem.

   Let $G$ be a group and $H \trianglelefteq G$, $K \trianglelefteq G$ and $K \subseteq H$. Prove the following:

   i. $K \trianglelefteq H$;

   ii. $H/K \trianglelefteq G/K$;

   iii. $\dfrac{G/K}{H/K} \backsimeq G/H$.

   (b) Write down the class equation of a finite group $G$.

   Let $G$ be a group of order $p^n$, where $p$ is a prime number and $n$ is a positive integer. Prove the following:

   i. $Z(G)$ is non - trivial;

   ii. if $n = 2$, then the group $G$ is abelian.

   (State any result that you may use)

4. (a) Define commutator subgroup $G'$ of a group $G$.

   Prove the following:

   i. $G$ is abelian if and only if $G' = \{e\}$;

   ii. $G' \trianglelefteq G$.

   (b) Let $H$ and $K$ be subgroups of a finite group $G$ such that $H \cap K = \{e\}$. Prove that $O(HK) = O(H) \circ O(K)$, where $O(HK)$, $O(H)$ and $O(K)$ are order of $HK$, $H$ and $K$ respectively.

   (c) Prove that $N$ is a normal subgroup of a group $G$ if and only if $gNg^{-1} = N, \forall g \in G$.

   (d) Let $N$ be a normal subgroup of a group $G$. Prove that $NH = HN$, where $H$ is any subset of $G$.

5. Prove or disprove the following:

   (a) If all non-trivial subgroups of a group $G$ are cyclic, then $G$ is cyclic.

   (b) Every abelian group is cyclic.

   (c) The homomorphic image of an abelian group is abelian.

   (d) If $H$ and $K$ are two subgroups of a group $G$, then $H \cup K$ is a subgroup of $G$.

6. (a) Define the following terms as applied to a permutation group:

   i. cycle of order $r$;

   ii. transposition.

   iii. signature;

(b) Prove that the permutation group on $n$ symbols $S_n$ is a finite group of order $n!$.

Is $S_n$ abelian for $n > 2$? Justify your answer.

(c) Express the permutation $\sigma$ in $S_9$ as a product of disjoint cycles, where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 7 & 2 & 6 & 8 & 5 & 3 & 4 & 9 \end{pmatrix}.$$