EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
THIRD EXAMINATION IN SCIENCE - 2011/2012
FIRST SEMESTER (March, 2014)
MT 302 - COMPLEX ANALYSIS
(SPECIAL REPEAT)

Answer all Questions | Time: Three hours

Q1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$.

(b) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ neighbourhood of a point $z_0 = x_0 + y_0$. Suppose that the first - order partial derivatives of the functions u and v with respect to x and y exist everywhere in that heightbourhood and that they are continuous at (x_0, y_0). Prove that, if those partial derivatives satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}; \ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

at (x_0, y_0), then the derivative $f'(z_0)$ exists.

(c) (i) Define what is meant by the function $h : \mathbb{R}^2 \to \mathbb{R}$ being harmonic.

(ii) Suppose that the function $F(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. Show that the functions $u(x, y)$ and $v(x, y)$ are harmonic in D.

Q2. (a) (i) Define what is meant by a **path** $\gamma : [\alpha, \beta] \to \mathbb{C}$.

(ii) For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)dz$.

(b) Let $a \in \mathbb{C}, r > 0$and $n \in \mathbb{Z}$. Show that

$$\int_{C(a;r)} (z-a)^n dx = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1, \end{cases}$$

where $C(a;r)$ denotes a positively oriented circle with centre a and radius r. (State any results you use without proof).

(c) State the **Cauchy's Integral Formula.**

By using the **Cauchy's Integral Formula** compute the following integrals:

(i) $\displaystyle\int_{C(0;2)} \frac{z}{(9-z^2)(z+i)} dz$;

(ii) $\displaystyle\int_{C(0;1)} \frac{1}{(z-a)^k(z-b)} dz$; where $k \in \mathbb{Z}, |a| > 1$ and $|b| < 1$.

Q3. (a) State the **Mean Value Property for Analytic Functions.**

(b) (i) Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being **entire.**

(ii) Prove the Liouville's Theorem: If f is entire and

$$\frac{max\{|f(t)| : |t| = r\}}{r} \to 0, \text{ as } r \to \infty,$$

then f is constant.

(State nay results you use without proof).

Prove that a bounded entire function is constant.

(c) Prove the **Maximum - Modulus Theorem**: Let f be analytic in an open connected set A. Let γ be a simple closed path that is connected, together with its inside, in A. Let

$$M := \sup_{z\in\gamma} |f(z)|.$$

If there exists z_0 inside γ such that $|f(z_0)| = M$, then f is constant throughout A. Consequently, if f is not constant in A, then

$$|f(z)| < M, \ \forall z \text{ inside } \gamma.$$

(State any theorem you use without proof)

Q4. (a) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \to \mathbb{C}$, where $D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}$. Define what is meant by

(i) f having a singularity at z_0;

(ii) the order of f at z_0;

(iii) f having a pole or zero at z_0 of order m;

(iv) f having a simple pole or simple zero at z_0.

(b) Prove that an isolated singularity z_0 of f is removable if and only if f is bounded on some deleted neighborhood $D^*(z_0; \delta)$ of z_0.

(c) Prove that if f has a simple pole at z_0, then

$$Res(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z_0).$$

Q5. Let f be analytic in the upper - half plane $\{z : Im(z) \geq 0\}$, except at finitely many points, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leq \frac{M}{|z|^\alpha}, |z| \geq R \text{ with } Im(z) \geq 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x)dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of } f \text{ in the upper half plane.}$$

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} dx.$$

(You may assume without proof the Residue Theorem).

Q6. (a) State the **Argument Theorem.**

(b) Prove **Rouche's Theorem** : Let γ be a simple closed path in an open starset A. Suppose that

(i) f, g are analytic in A except for finitely many poles, none lying on γ.

(ii) f and $f + g$ have finitely many zeros in A.

(iii) $|g(z)| < |f(z)|, z \in \gamma$. Then

$$ZP(f + g; \gamma) = ZP(f; \gamma)$$

where $ZP(f + g; \gamma)$ and $ZP(f; \gamma)$ denote the number of zeros - number of poles inside γ of $f + g$ and f respectively, where each is counted as many times as its order.

(c) State the **Fundamental theorem of Algebra.**

(d) Prove that all 5 zeros of $P(z) = z^5 + 3z^3 + 1$ lie in $|z| < 2$.