



Answer all Questions

Time: Two hours

1. (a) With the usual notation, derive the continuity equation for a fluid flow in the form

$$\frac{d\rho}{dt} + \rho \nabla \cdot \underline{V} = 0.$$

- (b) In cartesian coordinates, establish the equation of continuity for an incompressible fluid in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where u, v and w are the cartesian components of the velocity.

Show that

$$\frac{k}{r^5} (3x^2 - r^2, 3xy, 3xz)$$

are the velocity components of a possible fluid motion and the motion is irrotational, where k is a constant and $r^2 = x^2 + y^2 + z^2$.

Hence determine the stream lines.

2. (a) With the usual notation, derive the *Euler's equation* for an incompressible and inviscid fluid flow.

Hence show that if the fluid flow is steady the *Euler's equation* can be written as

$$(\underline{v} \cdot \nabla) \underline{v} = \underline{F} - \frac{1}{\rho} \nabla p.$$

- (b) An incompressible and inviscid fluid obeying Boyle's law $p = k\rho$, where k is a constant, is in motion in a uniform tube of small section. Prove that if ρ be

the density of the fluid then the velocity v at a distance x at time t in the tube is given by the equation

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2} [(v^2 + k)\rho].$$

(c) State the *Kelvin circulation* theorem.

If the velocity field is given by $\underline{v} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ then calculate the circulation around a square with its corners at $(1, 0)$, $(2, 0)$, $(2, 1)$, $(1, 1)$.

3. (a) Let a gas occupy the region $r \leq R$, where R is a function of time t , and a liquid of constant density ρ lie outside the gas. By assuming that there is contact between the gas and the liquid all the time and that the motion is symmetric about the origin $r = 0$, show that the motion is irrotational.

If the velocity at $r = R$, the gas liquid boundary is continuous then show that the pressure p at a point $P(\underline{r}, t)$ in the liquid is given by

$$\frac{p}{\rho} + \frac{1}{2} \left(\frac{R^2 \dot{R}}{r^2} \right)^2 - \frac{1}{r} \frac{d}{dt} (R^2 \dot{R}) = f(t),$$

where $|\underline{r}| = r$ and dots denote differentiation with respect to time t .

- (b) Given that a liquid extends to infinity and is at rest there with constant pressure π . Prove that the gas and liquid interface pressure for a spherical bubble of radius R is

$$\pi + \frac{\rho}{2R^2} \frac{d}{dR} (R^3 \dot{R}^2).$$

If the gas obeys Boyle's law $pv^{1+\alpha} = \text{constant}$, (where α is a constant and v is the volume of the gas) and expands from rest at $R = a$ to a position of rest at $R = 2a$, deduce that the initial pressure is

$$\frac{7\alpha\pi}{1 - 2^{-3\alpha}}.$$

4. (a) With the usual notation, derive the *Bernoulli's equation*:

$$\int \frac{dp}{\rho} + \frac{1}{2} v^2 + \Omega = \text{constant}.$$

- (b) A long pipe is of length l and has slowly tapering cross section. It is inclined at angle α to the horizontal and water flows steadily through it from the upper to the lower end. The section at the upper end has twice the radius of the

lower end. At the lower end the water is delivered at atmospheric pressure π .
If the pressure at the upper end is twice atmospheric, Show that the velocity
of delivery is'

$$\left[\frac{32}{15} \left(gl \sin \alpha + \frac{\pi}{\rho} \right) \right]^{\frac{1}{2}}$$

