EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2008/2009

FIRST SEMESTER (July/Aug., 2015)

EXTMT 201 - VECTOR SPACES AND MATRICES

(EXTERNAL DEGREE)

REPEAT

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wer all questions — Time: Three hours

(a) Define the term *subspace of a vector space.*

Let V be a vector space over a field $\mathbb{F}$. Prove that a non-empty subset S of V is a subspace of V if and only if $\alpha x + \beta y \in S$, for any $x, y \in S$ and $\alpha, \beta \in \mathbb{F}$.

(b) Prove that, $V = \left\{ f \in C[a,b] : f\left(\frac{a+b}{2}\right) = 0 \right\}$, is a vector space with usual addition of functions and scalar multiplication, where the set $C[a,b]$ denotes the set of all real valued continuous functions defined in the interval $[a,b] \subseteq \mathbb{R}$.

Is $f\left(\frac{a+b}{2}\right) = 1$, a vector space under the same operations? Justify your answer.

(a) Define the following:

(i) a *linearly independent* set of vectors;

(ii) a *basis* for a vector space.

(b) Let V be an $n-$dimensional vector space.

Prove the following:

(i) A linearly independent set of vectors of V with n elements is a basis for V.

(ii) Any linearly independent set of vectors of V may be extended as a basis for V.

(iii) If L is a subspace of V, then there exists a subspace M of V s

$V = L \oplus M$, where $\oplus$ denote the direct sum.

(c) i. Extend the subset $\{(1, 2, -1, 1), (0, 1, 2, -1)\}$ to a basis for $\mathbb{R}^4$.

ii. Let V be a vector space over the field F. Suppose that $v_1, v_2, \cdots$ linearly dependent vectors of V such that $v_1, v_2, \cdots, v_{m-1}$ are linear pendent. Prove that $v_m \in \langle v_1, v_2, \cdots, v_{m-1} \rangle$.

3. (a) Define the *range space* $R(T)$ and the *null space* $N(T)$ of a linear transfo T from a vector space V into another vector space W.

Find $R(T), N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z), \forall (x, y, z) \in \mathbb{R}^3.$$

Verify the equation $dim\ V = dim(R(T)) + dim(N(T))$ for this linear tra tion.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x, y, z) = (x + 2y,\ x + y + z,\ z)$ be transformation and let $B_1 = \{(1, 1, 1),\ (1, 2, 3),\ (2, -1, 1)\}$ and $B_2 = \{(1, 1, 0),\ (0, 1, 1), (1, 0, 1)\}$ be bases of $\mathbb{R}^3$.

i. Find the matrix representation of T with respect to the basis B_1;

ii. Using the transition matrix, find the matrix representation of T wit to the basis B_2.

4. (a) Define the following terms:

(i) *rank* of a matrix;

(ii) *row reduced echelon form* of a matrix.

(b) Let A be an $m \times n$ matrix. Prove the following:

(i) row rank of A is equal to column rank of A;

(ii) if B is a matrix obtained by performing an elementary row operati then A and B have the same rank.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

Define the term *adjoint of* A as applied to an $n \times n$ matrix $A = (a_{ij})$.

(a) With the usual notations, prove that

$$A \cdot (adj A) = (adj A) \cdot A = det A \cdot I.$$

Hence find the inverse of the matrix

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 3 & -2 \\ 2 & -1 & 3 \end{bmatrix}.$$

(b) Prove that, if $A = \begin{pmatrix} 2x & -x^2 \\ 1 & 0 \end{pmatrix}$, then $A^n = \begin{pmatrix} (n+1)x^n & -nx^{n+1} \\ nx^{n-1} & (1-n)x^n \end{pmatrix}$, where $x \in \mathbb{N}$.

(c) By applying appropriate row(column) operations, prove that the determinant of the matrix

$$\begin{pmatrix} 1+x_1 & 1 & 1 \\ 1 & 1+x_2 & 1 \\ 1 & 1 & 1+x_3 \end{pmatrix}$$

can be expressed as $x_1x_2x_3\left(1+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)$, where $x_1, x_2, x_3 \in \mathbb{R} \setminus \{0\}$.

6. (a) Let P be a n square matrix such that $P^2 = P$ and λ be a real number su

$\lambda \neq 1$, prove that $(I_n - \lambda P)$ is non-singular and that

$$(I_n - \lambda P)^{-1} = I_n + \frac{\lambda}{1-\lambda}P,$$

where I_n is the identity matrix of order n.

(b) State the necessary and sufficient condition for a system of linear equatio

consistent.

Show that the system of equations

$$\begin{aligned} x_1 - 3x_2 + x_3 + cx_4 &= b \\ x_1 - 2x_2 + (c-1)x_3 - x_4 &= 2 \\ 2x_1 - 5x_2 + (2-c)x_3 + (c-1)x_4 &= 3b+4 \end{aligned}$$

is consistent, for all values of b if $c \neq 1$. Find the value of b for which the

is consistent if $c = 1$ and obtain the general solution for these values.

(c) State Crammer's rule for 3×3 matrix and use it to solve

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ 2x_1 - 3x_2 - 2x_3 &= 1. \end{aligned}$$