



EASTERN UNIVERSITY, SRI LANKA
EXTERNAL FIRST YEAR EXAMINATION IN SCIENCE

2007/2008

SECOND SEMESTER

(June/September - 2015)

Repeat

MT 104 - DIFFERENTIAL EQUATIONS

AND

FOURIER SERIES

Answer All Questions

Time Allowed: 3 Hours

- Q1. (a) State the necessary and sufficient condition for the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

to be exact.

[10 Marks]

Hence solve the following differential equation

$$(x\sqrt{x^2 + y^2} - y) dx + (y\sqrt{x^2 + y^2} - x) dy = 0.$$

[20 Marks]

- (b) By differentiating the equation

$$\int \frac{f(xy) + F(xy)}{f(xy) - F(xy)} \frac{d(xy)}{xy} + \log \frac{x}{y} = C,$$

where C is a constant, verify that $\frac{1}{xy\{f(xy) - F(xy)\}}$ is an integrating factor of the differential equation

$$yf(xy) dx + xF(xy) dy = 0.$$

[20 Marks]

Hence solve the differential equation

$$y(1 + 2xy) dx + x(1 - xy) dy = 0.$$

[20 Marks]

- (c) If $y_1 = 2x$ is a particular solution of the following non-linear Riccati differential equation

$$\frac{dy}{dx} = 2 - 2xy + y^2,$$

obtain the general solution of the differential equation .

[30 Marks]

- Q2. (a) If $F(D) = \sum_{i=0}^n p_i D^i$, where $D \equiv \frac{d}{dx}$ and $p_i, i = 1, \dots, n$, are constants with $p_0 \neq 0$, prove the following formulas:

(i) $\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}$, where α is a constant and $F(\alpha) \neq 0$;

(ii) $\frac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D + \alpha)} V$, where V is a function of x .

[40 Marks]

- (b) Find the general solution of the following differential equations by using the results in (a).

(i) $(D^4 - 2D^2 + 1)y = 40 \cosh x$.

(ii) $(D^3 - 3D^2 - 6D + 8)y = xe^{-3x}$.

[60 Marks]

- Q3. (a) Let $x = e^t$. Show that

$$x \frac{d}{dx} \equiv \mathcal{D}, \quad x^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D},$$

and

$$x^3 \frac{d^3}{dx^3} \equiv \mathcal{D}(\mathcal{D} - 1)(\mathcal{D} - 2),$$

where $\mathcal{D} \equiv \frac{d}{dt}$. [20 Marks]

Use the above results to find the general solution of the following differential equation

$$x^3 y''' + xy' - y = 3x^4, \quad \text{where } ' \equiv \frac{d}{dx}.$$

[30 Marks]

(b) With $D \equiv \frac{d}{dt}$, solve the following simultaneous differential equations

$$D^2 x - \alpha^2 y = 0,$$

$$D^2 y + \alpha^2 x = 0.$$

[50 Marks]

Q4. Use the method of Frobenius to find the general solution of

$$(x - 1)^2 \frac{d^2 y}{dx^2} + (3x^2 - 4x + 1) \frac{dy}{dx} - 2y = 0$$

by expanding about $x = 1$. [100 Marks]

Q5. (a) Solve the following system of differential equations:

$$(i) \frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)};$$

$$(ii) \frac{dx}{x(y^3 - 2x^3)} = \frac{dy}{y(2y^3 - x^3)} = \frac{dz}{9z(x^3 - y^3)}.$$

[30 Marks]

- (b) Write down the condition of integrability of the total differential equation

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = 0.$$

[5 Marks]

Hence solve the following equation

$$(yz + xyz) dx + (zx + xyz) dy + (xy + xyz) dz = 0.$$

[15 Marks]

- (c) Find the general solution of the following linear first-order partial differential equations:

(i) $(y - z)p + (z - x)q = y - x;$

(ii) $(x^2 + y^2 - yz)p - (x^2 + y^2 - xz)q = z(x - y);$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}.$

[30 Marks]

- (d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$p(q^2 + 1) + (b - z)q = 0.$$

Here, $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}.$

[20 Marks]

- Q6. (a) Prove that if $-\pi \leq x \leq \pi$ and a is not an integer, then

$$\cos ax = \frac{2a \sin a\pi}{\pi} \left\{ \frac{1}{2a^2} - \frac{\cos x}{a^2 - 1} + \frac{\cos 2x}{a^2 - 4} - \dots \right\}.$$

[20 Marks]

Use the above result to show that

$$\frac{a\pi}{\sin a\pi} = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n a^2}{a^2 - n^2}.$$

[20 Marks]

(b) Use Fourier transform to solve the one-dimensional heat equation

$$\frac{\partial U}{\partial t} = 2 \frac{\partial^2 U}{\partial x^2} = 0,$$

subject to the boundary conditions

$$U(0, t) = 0, \quad U(x, 0) = e^{-x}, \quad x > 0$$

and $U(x, t)$ is bounded where $x > 0$ and $t > 0$.

[50 Marks]

(c) Prove the following identities for Bessel functions:

(i) $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$, $\nu \geq 1$;

(ii) $J'_\nu - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x)$.

[10 Marks]