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EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2008/2009

FIRST SEMESTER(June/July'2012)

EXTERNAL DEGREE

EXTMT 201 - VECTOR SPACES AND MATRICES

Answer all question | Time: Three hours

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1. Define what is meant by a subspace of a vector space.

   (a) Let $V = \left\{ \sum_{i=0}^{n} a_i X^i \ : \ a_i \in \mathbb{R}, n \in \mathbb{N} \right\}$ be the set of polynomials in one variable with real coefficients. The vector addition and scalar multiplication are defined as follows

   $$\left(\sum_{i=0}^{m} a_i X^i \right) + \left(\sum_{j=0}^{n} b_j X^j \right) = \sum_{r} (a_r + b_r) X^r$$

   where $a_r = 0$, if $r > m$ and $b_r = 0$, if $r > n$, and

   $$\alpha . \left(\sum_{i=0}^{n} a_i X^i \right) = \sum_{i=0}^{n} \alpha a_i X^i, \ \forall \ \alpha \in \mathbb{R}.$$

   Prove that V is a vector space over the field $\mathbb{R}$.

   (b) Let W_1 and W_2 be two subspaces of a vector space V over the field F and let A_1 and A_2 be two non-empty subsets of V. Prove with the usual notations that

   i. $W_1 + W_2 = \langle W_1 \cup W_2 \rangle$;

   ii. if $\langle A_1 \rangle = W_1$ and $\langle A_2 \rangle = W_2$ then $\langle A_1 \cup A_2 \rangle = W_1 + W_2$.

2. (a) Define the following terms:

   i. a linearly independent set of vectors;

   ii. a basis of a vector space.

(b) Prove that the non-zero vectors $v_1, v_2, \cdots, v_n$ of a vector space V over the F are linearly dependent if and only if one of them say $v_i, i \in \{2, 3, \cdots, n$ a linear combination of the preceding vectors.

(c) i. State and prove the dimension theorem for two subspaces of a finit mensional vector space.

   ii. Let $U = \langle\{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\}\rangle$ and $W = \langle\{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\}\rangle$.
   Find $dim(U + W)$ and $dim(U \cap W)$.

3. (a) Let T be a linear transformation from a vector space V into another ve space W. Define:

   i. range space $R(T)$;

   ii. null space $N(T)$.

   Find $R(T), N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined b

$$T(x, y, z) = (x - y + 2z, 2x + y, -x - 2y + 2z), \forall (x, y, z) \in \mathbb{R}^3.$$

   Verify the equation $dim\ V = dim(R(T)) + dim(N(T))$ for this linear t formation.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y) = (x + 2y, 2x - y, -x)$ and let $B_1 = \{(0, 1), (1, 1)\}$ and $B_2 = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ be bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Find the matrix representation of T with respect to the basis B_1 and with respect to the basis B_2 by using the transition matrix.

4. (a) Define the following terms as applied to a matrix:

   i. rank;

   ii. echelon form;

   iii. row reduced echelon form.

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(b) Let A be an $n \times n$ matrix. Prove that:

   i. row rank of A is equal to column rank of A;

   ii. if B is an $n \times n$ matrix, obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

(c) i. Find the row rank of the matrix $\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}$.

   ii. Find the row reduced echelon form of the matrix $\begin{pmatrix} -1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}$.

5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$:

   i. cofactor A_{ij} of an element a_{ij};

   ii. adjoint of A.

   Prove with the usual notations that

$$A.(adj\ A) = (adj\ A).A = det\ A.\ I$$

(b) i. if $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, then use the mathematical induction to prove

$$A^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix},$$ for every positive integer n.

ii. show that $\det A = (x-y)(x-z)(x-w)(y-z)(y-w)(z-w)$ for

$$A = \begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{pmatrix}.$$

6. (a) State the necessary and sufficient condition for a system of linear equati[ons to] be consistent.

Reduce the augmented matrix of the following system of linear equati[ons to] its row reduced echelon form and hence determine the conditions on no[n-zero] scalars $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 such that the system has

(i) a unique solution;

(ii) no solution;

(iii) more than one solution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

(b) Using row reduced echelon form, check wether the following system of equa[tions] is consistent, if so, find the solution(s).

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 &= -4 \\ 4x_1 - 2x_2 + x_3 + 2x_4 &= 20 \\ 3x_1 - x_2 + 3x_3 - 2x_4 &= 18 \\ 5x_1 - 3x_2 + 4x_3 - 3x_4 &= 32. \end{aligned}$$

(c) Solve the following system of linear equations by using Crammer's rule.

$$\begin{aligned} x + 2y + 3z &= 10 \\ 2x - 3y + z &= 1 \\ 3x + y - 2z &= 9. \end{aligned}$$