EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE - 2009/201

THIRD YEAR, FIRST SEMESTER(JUNE/SEPT., 2012)

EXTMT 302- COMPLEX ANALYSIS

(PROPER)

Answer all Questions — Time: Three hours

Q1. (a) Define what is meant by a complex-valued function f, defined on a domain $D(\subseteq \mathbb{C})$, has a limit at $z_0 \in D$.

(i) Prove that if a complex-valued function f has a limit at $z_0 \in D$, then it is unique.

(ii) Show that

$$\lim_{z \to i} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = 4 + 4i.$$

(b) (i) Let $f : S \subseteq \mathbb{C} \to \mathbb{C}$ and let z_0 be an interior point of S. Define what is meant by f being continuous at z_0 and on S.

Show that the function

$$f(z) = z^2$$

is continuous at $z = z_0$.

(ii) Is the function

$$f(z) = \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$$

continuous at $z = i$? Justify your answer.

Q2. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by f bein analytic at $z_0 \in A$.

(b) Let the function$f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ neigl borhood of a point $z_0 = x_0 + y_0$. Suppose that the first-order partial derivativ of the functions u and v with respect to x and y exist everywhere in that neigl borhood and that they are continuous at (x_0, y_0). Prove that, if those parti derivatives satisfy the Cauchy-Riemann equations. $u_x = v_y$ and $u_y = -v_x$ a (x_0, y_0), then the derivative $f'(z_0)$ exists.

(c) (i) Show that, if $f(z) = u(x, y)+iv(x, y)$ is analytic in a region S and $f'(z) =$ everywhere in S. Then f is constant throughout S.

(ii) Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a region S. Show that tl component functions u and v are harmonic in S.

Q3. (a) (i) Define what is meant by a path $\gamma : [\alpha, \beta] \to \mathbb{C}$.

(ii) For a path γ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)dz$.

(b) Let $a \in \mathbb{C}$, $r > 0$ and $n \in \mathbb{Z}$. Show that

$$\int_{C(a;\ r)} (z - a)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}.$$

(c) State the Cauchy's Integral Formula.

By using the Cauchy's Integral Formula compute the following integrals:

(i) $\displaystyle\int_{C(0;2)} \frac{z}{9 - z^2} dz;$

(ii) $\displaystyle\int_{C(0;1)} \frac{1}{(z - a)^k (z - b)} dz$, where $k \in \mathbb{Z}$, $|a| > 1$ and $|b| < 1$.

Q4. (a) State the Mean Value Property for Analytic Functions.

(b) (i) Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being entire.

(ii) Prove the Liouville's Theorem: If f is entire and bounded then f is co stant.

(State any results you use without proof).

Suppose that the function $J(z) = u(x, y) + iv(x, y)$ is analytic everywh in the $xy-$plane. Prove that $u(x, y)$ is constant throughout the plane.

(c) Prove the Maximum-Modulus Theorem: Let f be analytic in an open connected set A. Let γ be a simple closed path that is connected, together with its inside, in A. Let

$$M := \sup_{z\in\gamma} |f(z)|.$$

If there exists z_0 inside γ such that $|f(z_0)| = M$, then f is constant throughout A. Consequently, if f is not constant in A, then

$$|f(z)| < M, \ \forall z \text{ inside } \gamma.$$

(State any theorem you use without proof)

Q5. (a) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \to \mathbb{C}$, where $D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}$. Define what is meant by

i. f having a singularity at z_0;

ii. the order of f at z_0;

iii. f having a pole or zero at z_0 of order m;

iv. f having a simple pole or simple zero at z_0.

(b) Prove that

$$\text{ord}(f; z_0) = m \text{ if and only if } f(z) = (z - z_0)^m g(z), \ \forall\, z \in D^*(z_0; \delta),$$

for some $\delta > 0$, where g is analytic in $D(z_0; \delta)$ and $g(z_0) \neq 0$.

(c) Prove that if f has a simple pole at z_0, then

$$Res(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z_0).$$

Q6. Let f be analytic in $\{z : Im(z) \geq 0\}$, except possibly for finitely many singularities none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leq \frac{M}{|z|^{\alpha}}, \; |z| \geq R$$

with $Im(z) \geq 0$.

Then prove that

$$I := \int_{-\infty}^{\infty} f(x)dx$$

converges (exists) and $I = 2\pi i \times$ Sum of Residues of f in the upper half plane.

Hence evaluate the following integrals :

i. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$;

ii. $\displaystyle\frac{1}{2\pi i} \int_{C(0;3)} \frac{e^{zt}}{z^2(z^2+2z+2)} dz.$