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Predicting three-dimensional fractal dimensions of electrical discharges using two-dimensional projections

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Abstract

Three dimensional electrical discharge patterns are simulated to compare the three-dimensional structure with the two-dimensional projections of the same. The discharge patterns are simulated using a stochastic dielectric breakdown model. Fractal techniques are used to characterize the morphological structures of the simulated discharge patterns. The discharge patterns are simulated on a 50×50×50 cubic lattice and the fractal dimensions in both three-dimensions and two-dimensions are estimated using the Sandbox fractal dimension estimation method. The complexity of the simulated electrical tree patterns is strongly dependent on the exponent of the cell potential 'η'. When the value of the exponent was increased, the growth patterns effectively lost their fractal structure and became a curve with dimension 1. The fractal dimension of the three-dimensional growth patterns and two-dimensional projections when the exponent is close to unity (η≈1) are 1.84 and 1.53 respectively. A strong linear correlation was found between the simulated three-dimensional structures and their two-dimensional projections when the dimension is less than 2. This relation can be used to estimate the three-dimensional fractal dimension from two-dimensional views of complex electrical discharges.

Keywords: *Dielectric Breakdown Model, Electrical discharges, Fractal dimension, Lightning.*

INTRODUCTION

To simulate electric breakdown patterns such as lightning discharges in atmosphere and particulate aggregates formed in coagulation processes, two models are widely used, namely, the dielectric breakdown model (DBM) and the diffusion limited aggregation (DLA) model. Although there is a one to one correspondence between the DBM and the DLA when the structure size is very large [1], the DBM provides a better theoretical basis to simulate electrical breakdowns than the DLA model. Many published results are available in the literature on modelling the dielectric breakdown in two dimensions using DBM [2-5]. The same model has been extended to simulate three dimensional lightning patterns by a few authors with promising results [6-8].

The original stochastic DBM was developed by Niemeyer *et. al.* [2] to describe the geometric structure of the dielectric breakdown of surface discharges. The model was developed as a stochastic growth pattern, which finds the new growth sites according to the potential in the nearest neighbours by solving the discrete Laplace equation. The basic assumption is that the growth probability of conducting structures depends on the local electric field. They have shown that stochastic models for dielectric breakdown naturally lead to fractal structures. They reported the average fractal dimension of simulated surface discharge patterns as 1.75 ± 0.02.

Barclay *et. al.* [6] used a modified model of DBM [9] to simulate the electrical treeing in solid dielectrics. Their modified model had a point plane geometry and a different environment for the stepwise propagation of damage structures known as electrical trees under alternative voltage excitations. They have extended their study from electrical tree growth in two dimensions to three dimensions to check whether the patterns of behaviour established for the planar simulation remain valid for simulation in a more realistic geometry such as three dimensional volumes. They reported the two dimensional fractal dimension as 1.66 and a typical three dimensional fractal dimension as 2.5. Structures with dimension 1.7 appropriate for branch type electrical trees were seen only for higher values of the model parameter 'η' which is the exponent in the breakdown probability distribution.

Sanudo *et. al.* [7] extended the original two dimensional DBM model to three dimensions to simulate dielectric breakdown in the atmosphere in order to calculate the fractal dimension of simulated lightning discharges. The effects due to the finite size of the lattice have been tested by simulating the discharge structures in two cubic lattice sizes $60\times100\times100$ and $30\times50\times50$. They reported the fractal dimension in three dimensions as 1.51 and the vertical projections in two dimensions as 1.34 for higher value of the exponent ($\eta\approx6$).

In a more recent work [8], both experimental and simulation studies on fractal analysis of electrical trees, have been reviewed. For highly dense patterns, one cannot obtain the precise value of the fractal dimension directly from the two dimensional projected patterns, because part of the three dimensional structure will be covered when projected. The relationship between the fractal dimension of the projected patterns and real trees indicates that three dimensional fractal dimension deviates substantially from the two dimensional projections when the three dimensional fractal dimension is greater than 2. This review indicates that quantitative and systematic analyses of reconstructed patterns in three dimensions have not yet been published in the literature.

The main purpose of this work is to simulate the dielectric breakdown in the atmosphere and predict the three dimensional fractal dimensions from two dimensional projections. In particular, we are interested in quantifying the relationship between the three dimensional fractal dimension and two dimensional projections at lower values of fractal dimension (fractal dimension < 2) which is applicable to natural lightning discharges.

The structure of this paper is as follows. In section 2, the details of the three dimensional dielectric breakdown models and estimation of the fractal dimension using the Sandbox technique is given. In section 3, a discussion of the results obtained for three dimensional fractal dimensions together with the fractal dimension of the two dimensional projections are presented. In section 4, final conclusions are presented.

METHODOLOGY

Three Dimensional Dielectric Breakdown Model

In this work, the two-dimensional stochastic DBM [2] has been extended to simulate three-dimensional lightning discharge patterns within a 50×50×50 cubic lattice. The initial boundary condition for the three-dimensional growth structure is shown in Figure 1. The grey dot represents the cell having negative potential ($\phi=0$) at the top of the cubic lattice structure which is the starting point of the pattern (cloud base). The black layer represents the diffusion of the positive potential ($\phi=1$) cells at the bottom plane of the cubic lattice (ground plane). The gap between the point of the negative potential and the positive potential plane is equal to 50 lattice units.

The pattern develops stepwise from negative potential to positive potential depending on the magnitude of the cell potential at each lattice point. The magnitude of the cell potential at each lattice point can be calculated by solving the Laplace equation $\nabla^2\phi=0$, numerically over the three-dimensional grid, subjected to the initial boundary conditions.

The finite difference method can be used to calculate the numerical solution of the three-dimensional Laplace equation [10]. By using the nearest neighbours in the *x*-direction, the *y*-direction and the *z*-direction, potential at any arbitrary cell (*x*,*y*,*z*) can be calculated using the formula (1) given below for a three-dimensional lattice.

$$\phi(x,y,z)=\frac{1}{6}[\phi(i+1,j,k)+\phi(i-1,j,k)+\phi(i,j+1,k)+\phi(i,j-1,k)+\phi(i,j,k+1)+\phi(i,j,k-1)]$$

……………….. (1)

Once the potential ϕ is calculated for every lattice point, all the nearest neighbours are considered as possible candidate sites to extend the growth pattern. In Figure 1, the open circles (5 cells) show the selected cell configuration for the first step. Cell configuration is a very important factor when simulating three dimensional or two dimensional patterns. The standard cell configuration without diagonal points was used in this study in order to compare with the published results. It should be noted that the selection of diagonal points as nearest neighbours leads to lowering the dimension of the growth pattern. The probability of a cell being selected is governed by the probability expression (2) given below [7].

$$P_i=\frac{\phi_i^{\eta}}{\sum_{i=1}^{n}\phi_i^{\eta}} \qquad (i=1,\ldots\ldots n)$$

………………….(2)

where *n* is the number of candidate sites in each step and 'η' is the exponent. The selected cell is added to the pattern with the same potential as the potential of the growing pattern (ϕ=0). After step 1, the cell potential of each lattice point is recalculated (excluding the points that are already attached to the growing pattern) to introduce the effect due to the newly added cell on the whole system. For the 2nd step, there is a maximum of 9 possible cells to choose from, i.e., 5 new cells surrounding the newly added cell and 4 cells from the previous step (step 1) which were not chosen previously. In step 2, the DBM algorithm searches the next possible growth point out of all these possible candidate cells. At each step, one point is added to the growing pattern proportional to the cell probability.

The exponent 'η' is a model parameter that modulates the randomness of the process and it describes the relationship between the local field and the probability. Several investigations have been carried out to identify the behaviour of 'η'. Published results show that 'η' controls the appearance of the branches (complexity) in patterns [7].

Estimating the fractal dimension

The Sandbox method [6] which operates on the completed structure was used to calculate the 3D fractal dimension of the three dimensional structures and the 2D fractal dimension of the projections of the three dimensional structures. For 3D fractal dimension estimation using the Sandbox method, a box of size '*L*' is formed on the pattern and the mass of the pattern found within the box is evaluated. For 2D fractal dimension estimation, a square of size '*L*' is formed on the pattern and the mass of the pattern found within the square is evaluated. The mass can be evaluated by counting the number of lattice points within the box or the square. The average mass *M(L)* is obtained by placing the box (or the square) centred at each lattice point which is a member of the pattern and counting the lattice points within the box (or the square). This is repeated for different box (or square) sizes '*L*'. The fractal dimension *D* is the exponent that expresses the scaling of the mass with its size as shown below in (3).

$$M(L) \propto L^{D} \qquad \text{……………….. (3)}$$

In general, the fractal dimension can be found by fitting a straight line to a double log (log-log) plot and estimating the gradient. It should be noted that the fractal behaviour is valid only over a certain range for any given pattern (the lower bound being the cell size and the upper bound being the size of the pattern).

RESULTS

Examples of simulated three dimensional lightning discharges and their two dimensional projections on the XZ and the YZ planes are shown in Figure 2 for two values of the exponent, η=2 and η=10. The effect of the model parameter η can be seen clearly in the simulated patterns. High values of η limit the development of branches in patterns. The relationship between the three dimensional structure and two dimensional projections is important since the morphological structures of three dimensional natural electrical discharges such as lightning discharges can be studied only through two dimensional photographic images.

The estimated Sandbox fractal dimension values provide quantitative evidence for studying the dependency of the 3D and 2D fractal dimension with the exponent. The estimated average fractal dimension values together with their statistical uncertainties are categorized under Table 1 for η=2, 4, 6, 8 and 10. The reduction in the dimension of simulated patterns when moving from low values to high values of the exponent η, agrees well with the published work for both three dimensional and two dimensional simulated patterns [6, 7].

Table 1: 3D and 2D fractal dimension values for different η values.

Fractal Dimension	η=2	η=4	η=6	η=8	η=10
3D	1.66±0.02	1.59±0.11	1.39±0.06	1.14±0.10	1.05±0.08
2D – XZ plane	1.42±0.04	1.33±0.05	1.26±0.04	1.06±0.06	1.00±0.05
2D – YZ plane	1.44±0.02	1.39±0.07	1.21±0.08	1.05±0.06	1.01±0.06

Table 1 shows that, estimated fractal dimension values of two dimensional projections on the XZ and YZ planes are equal within their statistical uncertainties. This result is expected since there is no preferred direction in the development of the simulated patterns in an isotropic environment. The same result can be expected for the development of complex electrical discharges in an isotropic medium such as lightning discharges. i.e., complexity of the two dimensional view of a natural lightning discharge is independent of the position of the observer. The same result has been obtained by Barclay *et al*. in their simulation work [6]. When comparing the fractal dimension of the three dimensional simulated patterns and their two dimensional projections, a difference for the same η value is seen. The reader should note that there is an asymmetry between the vertical plane projections (XZ and YZ) and the horizontal plane projections (XY). However, we focus this work only on vertical plane projections since lightning is generally observed through photographic images which can be considered as vertical plane projects for all practical purposes.

The relationship between the fractal dimensions of the three dimensional structures and two dimensional projections is shown in Figure 3 for fractal dimensions less than 2. When estimating the two dimensional projections, the average of the XZ and the YZ projections were taken to reduce the statistical fluctuations. The 3D and 2D fractal dimensions show a strong linear correlation with a squared correlation coefficient of 0.998. The relationship between the two can be expressed by the following expression (4).

$$D_{3D} = 1.447 D_{2D} - 0.395 \qquad \text{.................... (4)}$$

The experimentally observed value for the average fractal dimension of lightning discharges is 1.34 [7]. This corresponds to a dimension of 1.54 in three dimensions which is approximately equal to the configuration when the exponent η=4. Published results indicate that for fractal dimensions greater than 2, fractal dimensions of two dimensional projections tend to disagree substantially with fractal dimensions of three dimensional structures [8].

In Figure 4, the variation of 2D and 3D fractal dimensions with the exponent is shown. It can be seen that for higher values of the exponent (η>10), the three dimensional structures and two dimensional projections effectively lose their fractal nature and become a curve with dimension 1. As shown in the figure, within statistical uncertainties, dependency of both dimensions on η can be expressed by two simple exponential equations. In principle, the tortuous nature of lightning is controlled by η in the simulation related to potential build up in the atmosphere for natural lightning events.

DISCUSSION AND CONCLUSIONS

The DBM provides the basic theoretical foundation for simulating the stochastic breakdown of electrical discharges. Since the discrete lattice potentials mimic the possible electric field effects generated by the space charges in the atmosphere, the DBM is a more realistic model to simulate long electrical breakdowns than the DLA model. In this work, the stochastic DBM, known as the η model was extended to simulate electrical discharges in three dimensions. The fractal dimension was used to characterize the complexity of simulated discharge patterns in three dimensions as well as two dimensional projections of the same.

Three dimensional lightning patterns with dimension 1.54 is equivalent to plane projections with dimension 1.34. Comparison with published values of fractal dimension of natural lightning discharges which were carried out through two dimensional photographic images indicate that, this is the corresponding dimension for simulating natural lightning discharges. At this dimension, the corresponding value of the exponent in the three dimensional simulation was η=4. For fractal dimensions less than 2, the three dimensional patterns and two dimensional projections are linearly correlated. This indicates that at low dimensions, there is no significant effect due to screening from the third dimension when two dimensional projections are taken. Since the dimensions reported for natural lightning discharges are below this limit, one can safely measure the dimension of natural discharges in 2D and estimate the corresponding dimension in 3D.

It would be interesting to study whether there is any difference between the fractal nature of lightning discharges in different regions in the world. Simulation indicates that the exponent 'η' which is related to the strength of the potential field, controls the probability of appearance of branches in the patterns. Thus, there could be a difference in the fractal nature of lightning between the tropical climate and the temperate climates. It is shown that when η>10, the patterns effectively lose their fractal structure and become a curve with dimension 1.

The major disadvantage in the three dimensional stochastic model discussed in this work is the execution time which is highly dependent on the size of the cubic lattice chosen. This is directly linked to the solving of Laplace equation iteratively for each step. Further work is in progress to develop techniques that can shorten the execution time.

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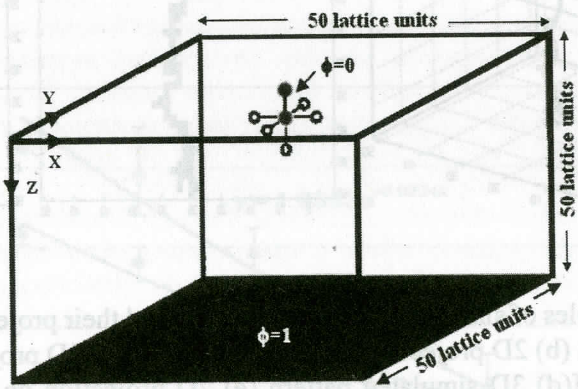

Figure 1: The initial boundary condition in a 50×50×50 cubic lattice. The discharge is assumed to be initiated downward at the top central point of the lattice. The open circles show the possible growth sites.

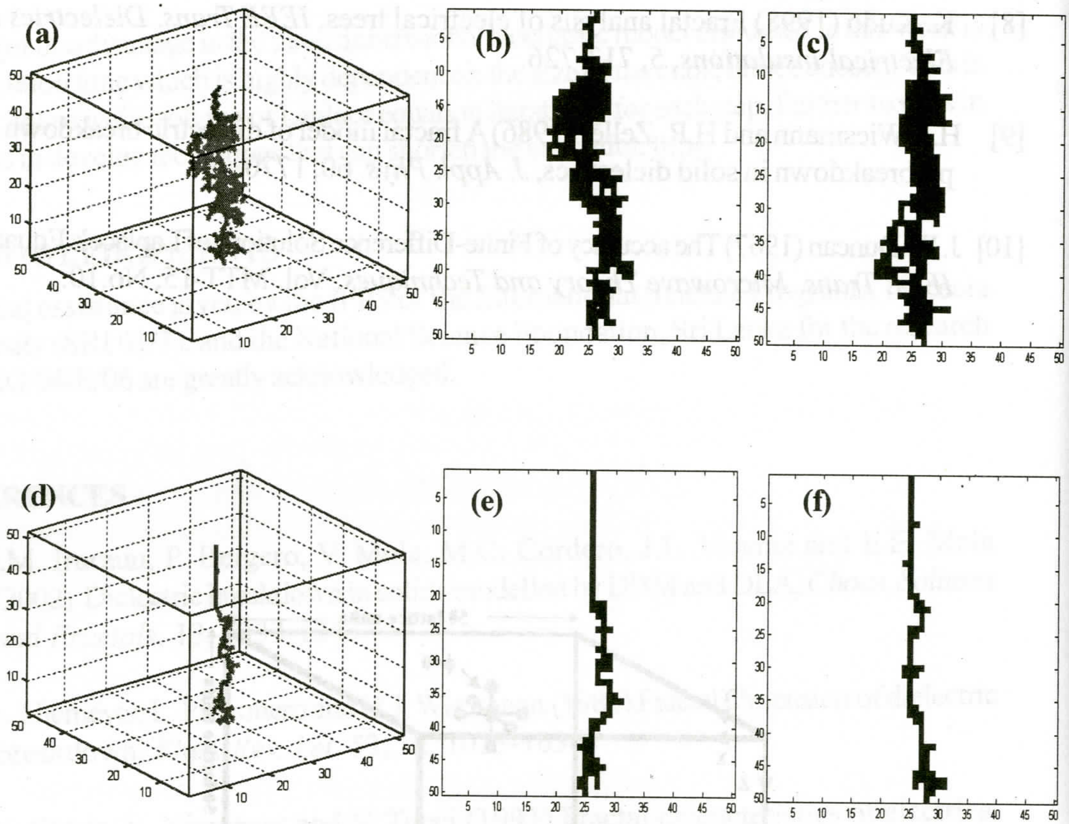

Figure 2: Examples of simulated lightning patterns and their projections when η=2 [(a) 3D simulated pattern (b) 2D projection on the XZ-plane & (c) 2D projection on the YZ-plane] and when η=10 [(d) 3D simulated pattern (e) 2D projection on the XZ-plane & (f) 2D projection on the YZ-plane].

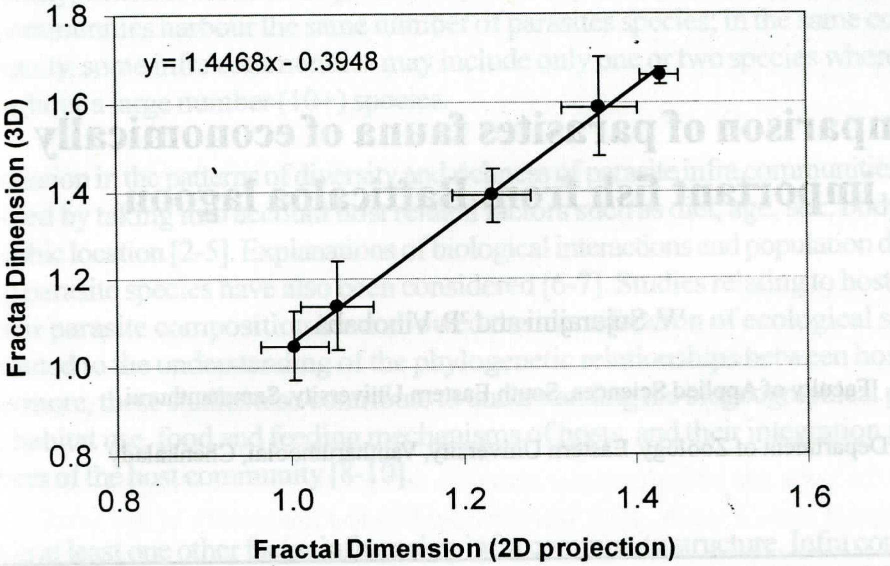

Figure 3: Correlation between the estimated fractal dimension of two dimensional projections and three dimensional structures.

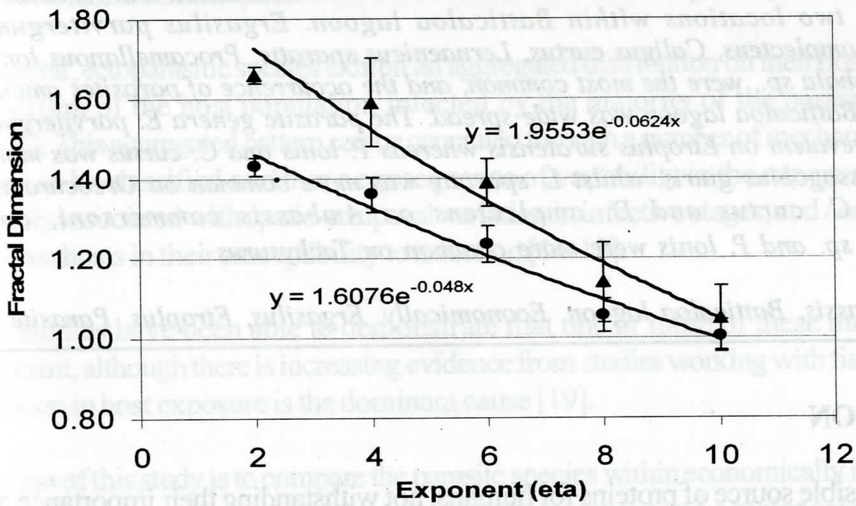

Figure 4: Variation of the estimated fractal dimension of three dimensional (triangles) structures and two dimensional projections (circles) with the exponent.