Charged relativistic spheres with generalized potentials

S. Thirukkanesh† and S. D. Maharaj

Astrophysics and Cosmology Research Unit,

School of Mathematical Sciences,

University of KwaZulu-Natal,

Private Bag X54001,

Durban 4000,

South Africa.

Abstract

A new class of exact solutions of the Einstein-Maxwell system is found in closed form. This is achieved by choosing a generalised form for one of the gravitational potentials and a particular form for the electric field intensity. For specific values of the parameters it is possible to write the new series solutions in terms of elementary functions. We regain well known physically reasonable models. A physical analysis indicates that the model may be used to describe a charged sphere. The influence of the electromagnetic field on the gravitational interaction is highlighted.

Keywords: Einstein-Maxwell equations; exact solutions; charged stars.

AMS classification nos: 83C15; 83C22; 85A99

†Permanent address: Department of Mathematics, Eastern University, Sri Lanka, Chenkalady, Sri Lanka.

1 Introduction

Solutions of the Einstein-Maxwell system of equations for static spherically symmetric interior spacetimes are necessary to describe charged compact objects in relativistic astrophysics where the gravitational field is strong as in the case of neutron stars. In the presence of an electromagnetic field, the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided: the gravitational attraction is counterbalanced by the Coulombian repulsive force in addition to the pressure gradient. The recent analyses of Ivanov [1] and Sharma *et al* [2] show that the presence of the electromagnetic field affects the values of redships, luminosities and maximum mass of a compact relativistic object. Patel and Koppar [3], Tikekar and Singh [4], Mukherjee [5] and Gupta and Kumar [6] demonstrated that it is possible to model charged neutron stars with high densities with acceptable bounds for the surface redshift, luminosity and total mass. Thomas *et al* [7], Tikekar and Thomas [8] and Paul and Tikekar [9] demonstrated that charged relativistic solutions may be applied to core-envelope models. The role of the electromagnetic field in describing the gravitational behaviour of stars composed of quark matter has been recently highlighted by Mak and Harko [10] and Komathiraj and Maharaj [11]. Therefore, the Einstein-Maxwell system for a charged star has attracted considerable attention in various physical investigations. We note that at present there is no unified theory of electromagnetism and gravitation which is generally accepted. In this paper we have used the approach of coupling the electromagnetic field tensor to the matter tensor in Einstein's equations such that Maxwell's equations are satisfied. We believe that the qualitative features generated in this charged model should yield results which are physically reasonable.

Our objective is to generate a new class of solutions to the Einstein-Maxwell system that satisfies the physical criteria: the gravitational potentials, electric field intensity and matter variables must be finite and continuous throughout the stellar interior. The speed of the sound must be less than the speed of the light, and ideally the solution should be stable with respect to radial perturbations. A barotropic equation of state, linking the isotropic pressure to the energy density, is often assumed to constrain the matter distribution. In addition to these conditions the interior solution must match smoothly at the boundary of the stellar object with the Reissner-Nordstrom exterior spacetime. In recent years researchers have attempted to introduce a systematic approach to finding solutions to the field equations. Maharaj and Leach [12] generalised the Tikekar superdense star, Thirukkanesh and Maharaj [13] generalised the Durgapal and Bannerji neutron star and Maharaj and Thirukkanesh [14] generalised the John and Maharaj [15] model. These new classes of models were obtained by reducing the condition of pressure isotropy to a recurrence relation with real and rational coefficients which could be solved by mathematical induction, leading to new mathematical and physical insights in the Einstein-Maxwell field equations. We attempt to perform a similar analysis here to the coupled Einstein-Maxwell equations for a general form of the gravitational potentials with charged matter. We find that the generalised condition of pressure isotropy leads to a new recurrence relation which can be solved in general.

In this paper, we seek new exact solutions to the Einstein-Maxwell field equations, using the systematic series analysis, which may be used to describe the interior relativistic sphere. Our objective is to obtain a general class of exact solutions which contains previously known models as particular cases. This approach produces a number of difference equations, which we demonstrate can be solved explicitly from first principles. We first express the Einstein-Maxwell system of equations for static spherically symmetric line element as an equivalent system using the Durgapal and Bannerji [16] transformation in Section 2. In Section 3, we choose particular forms for one of the gravitational potentials and the electric field intensity, which reduce the condition of pressure isotropy to a linear second order equation in the remaining gravitational potential. We integrate this generalised condition of isotropy equation using Frobenius method in Section 4. In general the solution will be given in terms of special functions. However elementary functions are regainable, and in Section 5, we find two category of solutions in terms of elementary functions by placing certain restriction on the parameters. In Section 6, we regain known charged Einstein-Maxwell models and uncharged Einstein models from our general class of models. In Section 7, we discuss the physical features, plot the matter variables and show that our models are physically reasonable. Finally we summarise the results found in this paper in Section 8.

2 The Field Equations

The gravitational field should be static and spherically symmetric for describing the internal structure of a dense compact relativistic sphere which is charged. For describing such a configuration, we utilise coordinates $(x^a) = (t, r, \theta, \phi)$, such that the generic form of the line element is given by

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

The Einstein field equations can be written in the form

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho \tag{2a}$$

$$-\frac{1}{r^2}\left(1-e^{-2\lambda}\right)+\frac{2\nu'}{r}e^{-2\lambda} = p \tag{2b}$$

$$e^{-2\lambda}\left(\nu''+\nu'^2+\frac{\nu'}{r}-\nu'\lambda'-\frac{\lambda'}{r}\right) = p \tag{2c}$$

for neutral perfect fluids. The energy density ρ and the pressure p are measured relative to the comoving fluid 4-velocity $u^a = e^{-\nu}\delta^a_0$ and primes denote differentiation with respect to the radial coordinate r. In the system (2a)-(2c), we are using units where the coupling constant $\frac{8\pi G}{c^4}=1$ and the speed of light $c=1$. This system of equations determines the behaviour of the gravitational field for a neutral perfect fluid source. A different but equivalent form of the field equations can be found if we introduce the transformation

$$x = Cr^2, \qquad Z(x) = e^{-2\lambda(r)}, \qquad A^2y^2(x) = e^{2\nu(r)} \tag{3}$$

so that the line element (1) becomes

$$ds^2 = -A^2y^2dt^2 + \frac{1}{4CxZ}dx^2 + \frac{x}{C}(d\theta^2+\sin^2\theta d\phi^2).$$

The parameters A and C are arbitrary constants in (3). Under the transformation (3), the system (2a)-(2c) has the equivalent form

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{4a}$$

$$4Z\frac{\dot{y}}{y}+\frac{Z-1}{x} = \frac{p}{C} \tag{4b}$$

$$4Zx^2\ddot{y}+2\dot{Z}x^2\dot{y}+(\dot{Z}x-Z+1)y = 0 \tag{4c}$$

where dots represent differentiation with respect to x.

In the presence of an electromagnetic field the system (4a)-(4c) has to be replaced by the Einstein-Maxwell system of equations. We generate the system

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C} \tag{5a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} - \frac{E^2}{2C} \tag{5b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2x}{C}\right)y = 0 \tag{5c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2 \tag{5d}$$

where E is the electric field intensity and σ is the charge density. This system of equations governs the behaviour of the gravitational field for a charged perfect fluid source. When $E = 0$ the Einstein-Maxwell equations (5a)-(5d) reduce to the uncharged Einstein equations (4a)-(4c).

3 Choosing Z and E

We seek solutions to the Einstein-Maxwell field equations (5a)-(5d) by making explicit choices for the gravitational potential Z and the electric field intensity E on physical grounds. The system (5a)-(5d) comprises four equations in six unknowns Z, y, ρ, p, E and σ. Equation (5c), called the generalised condition of pressure isotropy, is the master equation in the integration process. In this treatment we specify the gravitational potential Z and electric field intensity E, so that it is possible to integrate (5c). The explicit solution of the Einstein-Maxwell system (5a)-(5d) then follows. We make the particular choice

$$Z = \frac{1+ax}{1+bx} \tag{6}$$

where a and b are real constants. The function Z is regular at the centre and well behaved in the stellar interior for a wide range of values of a and b. It is important to note that the choice (6) for Z is is physically reasonable. This form for the potential Z contains special cases which correspond to neutron star models, eg. when $a = -\frac{1}{2}$ and $b = 1$ we regain the uncharged dense neutron star of Durgapal and Bannerji [16]. When a is arbitrary and $b = 1$ then Thirukkanesh and Maharaj [13] and Maharaj and Komathiraj [17] found charged solutions to the Einstein-Maxwell system. These solutions can be used to model a charged relativistic sphere with desirable physical properties. Consequently the general form (6) contains known physically acceptable uncharged and charged relativistic stars for particular values of a and b. We seek to study the Einstein-Maxwell system with the choice (6) in an attempt to find new solutions, and to show explicitly that cases found previously can be placed into our general class of models.

Upon substituting (6) in equation (5c) we obtain

$$4(1+ax)(1+bx)\ddot{y} + 2(a-b)\dot{y} + \left[b(b-a) - \frac{E^2(1+bx)^2}{Cx}\right] y = 0. \tag{7}$$

The differential equation (7) is difficult to solve in the above form; we first introduce a transformation to obtain a more convenient form. We let

$$1 + bx = X, \quad Y(X) = y(x), \quad b \neq 0. \tag{8}$$

With the help of (8), (7) becomes

$$4X\left[aX - (a-b)\right]\frac{d^2Y}{dX^2} + 2(a-b)\frac{dY}{dX} + \left[(b-a) - \frac{E^2X^2}{C(X-1)}\right] Y = 0 \tag{9}$$

in terms of the new dependent and independent variables Y and X respectively. The differential equation (9) may be integrated once the electric field E is given. A variety of choices for E is possible but only a few have the desirable features in the stellar interior. Note that the particular choice

$$E^2 = \frac{\alpha C(X-1)}{X^2} = \frac{\alpha Cbx}{(1+bx)^2} \tag{10}$$

where α is a constant has the advantage of simplifying (9). The electric field given in (10) vanishes at the centre of the star, and remains continuous and bounded for all interior points in the star. When $b = 1$ then E^2 reduces to the expression in the treatment of Maharaj and Komathiraj [17]. Thus the choice for E in (10) is physically reasonable in the study of the gravitational behaviour of charged stars. With the help of (10) we find that (9) becomes

$$4X\left[aX - (a-b)\right]\frac{d^2Y}{dX^2} + 2(a-b)\frac{dY}{dX} + \left[(b-a) - \alpha\right] Y = 0. \tag{11}$$

The differential equation (11) becomes

$$4X\left[aX - (a-b)\right]\frac{d^2Y}{dX^2} + 2(a-b)\frac{dY}{dX} + (b-a)Y = 0 \tag{12}$$

when $\alpha = 0$ and there is no charge.

4 Solutions

We need to integrate the master equation (11) to solve the Einstein-Maxwell system (5a)-(5d). Two categories of solution are possible when $a = b$ and $a \neq b$.

4.1 The Case $a = b$

When $a = b$ equation (11) becomes

$$X^2 \frac{d^2Y}{dX^2} - \frac{\alpha}{4a} Y = 0 \tag{13}$$

which is an Euler-Cauchy equation. The solution of (13) becomes

$$Y = \begin{cases} c_1(1+ax)^{(1+\sqrt{1+\alpha/a})/2} + c_2(1+ax)^{(1-\sqrt{1+\alpha/a})/2} & \text{if } a > 0, \\ \\ \sqrt{1+ax}\left[c_1 \sin\left(\sqrt{\frac{a+\alpha}{4a}} \ln(1+ax)\right)\right. \\ \left. +c_2 \cos\left(\sqrt{\frac{a+\alpha}{4a}} \ln(1+ax)\right)\right] & \text{if } a < 0, \end{cases} \tag{14}$$

where c_1 and c_2 are constants. From (5a) and (6) we observe that $\rho = -\frac{E^2}{2}$. We do not pursue this case to avoid negative energy densities. It is interesting to observe that when $a = b = 0$ then it is possible to generate an exact Einstein-Maxwell solution to (5a)-(5d), for a different choice of E^2, which contains the Einstein universe as pointed out by Komathiraj and Maharaj [18].

4.2 The Case $a \neq b$

Observe that it is not possible to express the general solution of the master equation (11) in terms of conventional elementary functions for all values of a, b $(a \neq b)$ and α. In general the solution can be written in terms of special functions. It is necessary to express the solution in a simple form so that it is possible to conduct a detailed physical analysis. Hence in this section we attempt to obtain a general solution to the differential equation (11) in series form. In a subsequent section we show that it is possible to find particular solutions in terms of algebraic functions and polynomials.

We can utilise the method of Frobenius about $X = 0$, since this is a regular singular point of the differential equation (11). We write the solution of the differential equation (11) in the series form

$$Y = \sum_{n=0}^{\infty} c_n X^{n+r}, \quad c_0 \neq 0 \tag{15}$$

where c_n are the coefficients of the series and r is a constant. For an acceptable solution we need to find the coefficients c_n as well as the parameter r. On substituting (15) in the differential equation (11) we have

$$2(a-b)c_0 r[-2(r-1)+1]X^{r-1} + \sum_{n=1}^{\infty} [2(a-b)c_{n+1}(n+r+1)[-2(n+r)+1]$$

$$+c_n \left[4a(n+r)(n+r-1) - (a-b+\alpha)\right]] X^{n+r} = 0. \tag{16}$$

For consistency the coefficients of the various powers of X must vanish in (16). Equating the coefficient of X^{r-1} in (16) to zero, we find

$$(a-b)c_0 r[2(r-1)-1]=0$$

which is the indicial equation. Since $c_0 \neq 0$ and $a \neq b$, we must have $r=0$ or $r=\frac{3}{2}$. Equating the coefficient of X^{n+r} in (16) to zero we obtain

$$c_{n+1}=\frac{4a(n+r)(n+r-1)-[a-b+\alpha]}{2(a-b)(n+1+r)[2(n+r)-1]}c_n, \quad n \geq 0 \tag{17}$$

The result (17) is the basic difference equation which determines the nature of the solution.

We can establish a general structure for all the coefficients by considering the leading terms. We note that the coefficients $c_1, c_2, c_3, \ldots$ can all be written in terms of the leading coefficient c_0, and this leads to the expression

$$c_{n+1}=\prod_{p=0}^{n}\frac{4a(p+r)(p+r-1)-(a-b+\alpha)}{2(a-b)(p+1+r)[2(p+r)-1]}c_0 \tag{18}$$

where the symbol $\prod$ denotes multiplication. It is also possible to establish the result (18) rigorously by using the principle of mathematical induction. We can now generate two linearly independent solutions from (15) and (18). For the parameter value $r=0$ we obtain the first solution

$$\begin{aligned}
Y_1 &= c_0\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{4ap(p-1)-(a-b+\alpha)}{2(a-b)(p+1)(2p-1)}X^{n+1}\right] \\
y_1 &= c_0\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{4ap(p-1)-(a-b+\alpha)}{2(a-b)(p+1)(2p-1)}(1+bx)^{n+1}\right].
\end{aligned} \tag{19}$$

For the parameter value $r=\frac{3}{2}$ we obtain the second solution

$$\begin{aligned}
Y_2 &= c_0 X^{\frac{3}{2}}\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{a(2p+3)(2p+1)-(a-b+\alpha)}{(a-b)(2p+5)(2p+2)}X^{n+1}\right] \\
y_2 &= c_0(1+bx)^{\frac{3}{2}}\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{a(2p+3)(2p+1)-(a-b+\alpha)}{(a-b)(2p+5)(2p+2)}(1+bx)^{n+1}\right].
\end{aligned} \tag{20}$$

Therefore the general solution to the differential equation (7), for the choice (10), is given by

$$y=a_1 y_1(x)+b_1 y_2(x) \tag{21}$$

where a_1 and b_1 are arbitrary constants and y_1 and y_2 are given by (19) and (20) respectively. It is clear that the quantities y_1 and y_2 are linearly independent functions. From (5a)-(5d) and (21) the general solution to the Einstein-Maxwell system can be written as

$$e^{2\lambda} = \frac{1+bx}{1+ax} \tag{22a}$$

$$e^{2\nu} = A^2 y^2 \tag{22b}$$

$$\frac{\rho}{C} = \frac{(b-a)(3+bx)}{(1+bx)^2} - \frac{\alpha bx}{2(1+bx)^2} \tag{22c}$$

$$\frac{p}{C} = 4\frac{(1+ax)}{(1+bx)}\frac{\dot{y}}{y} + \frac{(a-b)}{(1+bx)} + \frac{\alpha bx}{2(1+bx)^2} \tag{22d}$$

$$\frac{E^2}{C} = \frac{\alpha bx}{(1+bx)^2}. \tag{22e}$$

The result in (22a)-(22e) is a new solution to the Einstein-Maxwell field equations. Note that if we set $\alpha = 0$, (22a)-(22e) reduce to models for uncharged stars which may contain new solutions to the Einstein field equations (4a)-(4c).

5 Elementary Functions

The general solution (21) can be expressed in terms of polynomial and algebraic functions. This is possible in general because the series (19) and (20) terminate for restricted values of the parameters a, b and α so that elementary functions are possible. Consequently we obtain two sets of general solutions in terms of elementary functions, by determining the specific restriction on the quantity $a - b + \alpha$ for a terminating series. The elementary functions found using this method, can be written as polynomials and polynomials with algebraic functions. We provide the details of the process in the Appendix; here we present a summary of the results. In terms of the original variable x, the first category of solution can be written as

$$\begin{aligned} y = {} & \\ & d_1(1+ax)^{\frac{1}{2}}\left[1-(n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+bx)^i\right] \\ & +d_2\,(1+bx)^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i \frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}(1+bx)^i\right] \end{aligned} \tag{23}$$

for $a-b+\alpha = a(2n+3)(2n+1)$, where d_1 and d_2 are arbitrary constants. The second category of solutions can be written as

$$y = d_3(1+ax)^{\frac{1}{2}}(1+bx)^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}(1+bx)^i\right]$$
$$+d_4\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+bx)^i\right] \quad (24)$$

for $a-b+\alpha = 4an(n-1)$, where d_3 and d_4 are arbitrary constants. It is remarkable to observe that the solutions (23) and (24) are expressed completely in terms of elementary functions only. This does not happen often considering the nonlinearity of the gravitational interaction in the presence of charge. We have given our solutions in a simple form: this has the advantage of facilitating the analysis of the physical features of the stellar models. Observe that our approach has combined both the charged and uncharged cases for a relativistic star: when $\alpha = 0$ we obtain the solutions for the uncharged case directly.

6 Known Solutions

It is interesting to observe that we can regain a number of physically reasonable models from the general class of solutions found in this paper. These individual models can be generated from the general series solution (21) or the simplified elementary functions (23) and (24). We explicitly generate the following models.

6.1 Case I: Hansraj and Maharaj charged stars

For this case we set $a = 0, b = 1$ and $0 \leq \alpha < 1$. Then from (19) we find that

$$\begin{aligned}
y_1 &= c_0\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{-(1-\alpha)}{2(p+1)(2p-1)}(\sqrt{1+x})^{2n+2}\right] \\
&= c_0\left(\left[1-\frac{(\sqrt{(1-\alpha)(1+x)})^2}{2!}+\frac{(\sqrt{(1-\alpha)(1+x)})^4}{4!}\right.\right. \\
&\quad \left.-\frac{(\sqrt{(1-\alpha)(1+x)})^6}{6!}+...\right]+\sqrt{(1-\alpha)(1+x)}\left[\sqrt{(1-\alpha)(1+x)}\right. \\
&\quad \left.\left.-\frac{(\sqrt{(1-\alpha)(1+x)})^3}{3!}+\frac{(\sqrt{(1-\alpha)(1+x)})^5}{5!}-...\right]\right) \\
&= c_0\cos\sqrt{(1-\alpha)(1+x)}+c_0\sqrt{(1-\alpha)(1+x)}\sin\sqrt{(1-\alpha)(1+x)}.
\end{aligned}$$

Equation (20) gives the result

$$
\begin{aligned}
y_2 &= c_0(\sqrt{1+x})^3\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{-(1-\alpha)}{(2p+5)(2p+2)}(\sqrt{1+x})^{2n+2}\right] \\
&= \frac{3c_0}{(\sqrt{1-\alpha})^3}\left(\left[\sqrt{(1-\alpha)(1+x)}-\frac{(\sqrt{(1-\alpha)(1+x)})^3}{3!}\right.\right. \\
&\quad \left.+\frac{(\sqrt{(1-\alpha)(1+x)})^5}{5!}-...\right]-\sqrt{(1-\alpha)(1+x)}\left[1\right. \\
&\quad \left.\left.-\frac{(\sqrt{(1-\alpha)(1+x)})^2}{2!}+\frac{(\sqrt{(1-\alpha)(1+x)})^4}{4!}-...\right]\right) \\
&= \frac{3c_0}{(\sqrt{1-\alpha})^3}\left[\sin\sqrt{(1-\alpha)(1+x)}-\sqrt{(1-\alpha)(1+x)}\cos\sqrt{(1-\alpha)(1+x)}\right].
\end{aligned}
$$

Hence the general solution becomes

$$
\begin{aligned}
y &= \left[D_1-D_2\sqrt{(1-\alpha)(1+x)}\right]\cos\sqrt{(1-\alpha)(1+x)} \\
&\quad +\left[D_2+D_1\sqrt{(1-\alpha)(1+x)}\right]\sin\sqrt{(1-\alpha)(1+x)}
\end{aligned} \tag{25}
$$

where D_1 and D_2 are new arbitrary constants. The class of charged solutions (25) is the first category found by Hansraj and Maharaj [19].

When $a=0, b=1$ and $\alpha=1$ we easily obtain the result

$$y = a_1 + b_1(1+x)^{\frac{3}{2}} \tag{26}$$

from (21). This is the second category of the Hansraj-Maharaj charged solutions.

We now set $a=0, b=1$ and $\alpha>1$. Then from (19) we obtain

$$
\begin{aligned}
y_1 &= c_0\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{(\alpha-1)}{2(p+1)(2p-1)}(\sqrt{1+x})^{2n+2}\right] \\
&= c_0\left(\left[1+\frac{(\sqrt{(\alpha-1)(1+x)})^2}{2!}+\frac{(\sqrt{(\alpha-1)(1+x)})^4}{4!}\right.\right. \\
&\quad \left.+\frac{(\sqrt{(\alpha-1)(1+x)})^6}{6!}+...\right]-\sqrt{(\alpha-1)(1+x)}\left[\sqrt{(\alpha-1)(1+x)}\right. \\
&\quad \left.\left.+\frac{(\sqrt{(\alpha-1)(1+x)})^3}{3!}+\frac{(\sqrt{(\alpha-1)(1+x)})^5}{5!}+...\right]\right) \\
&= c_0\cosh\sqrt{(\alpha-1)(1+x)}-c_0\sqrt{(\alpha-1)(1+x)}\sinh\sqrt{(\alpha-1)(1+x)}.
\end{aligned}
$$

Equation (20) gives the result

$$
\begin{aligned}
y_2 &= c_0(\sqrt{1+x})^3\left[1+\sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{(\alpha-1)}{(2p+5)(2p+2)}(\sqrt{1+x})^{2n+2}\right] \\
&= \frac{-3c_0}{(\sqrt{\alpha-1})^3}\left(\left[\sqrt{(\alpha-1)(1+x)}+\frac{(\sqrt{(\alpha-1)(1+x)})^3}{3!}\right.\right. \\
&\quad \left.+\frac{(\sqrt{(\alpha-1)(1+x)})^5}{5!}+...\right]-\sqrt{(\alpha-1)(1+x)}\,[1 \\
&\quad \left.\left.+\frac{(\sqrt{(\alpha-1)(1+x)})^2}{2!}+\frac{(\sqrt{(\alpha-1)(1+x)})^4}{4!}+...\right]\right) \\
&= \frac{-3c_0}{(\sqrt{\alpha-1})^3}(\sinh\sqrt{(1-\alpha)(1+x)}-\sqrt{(1-\alpha)(1+x)}\cosh\sqrt{(1-\alpha)(1+x)}).
\end{aligned}
$$

Therefore, the general solution becomes

$$
\begin{aligned}
y &= \left[D_2-D_1\sqrt{(\alpha-1)(1+x)}\right]\sinh\sqrt{(\alpha-1)(1+x)} \\
&\quad +\left[D_1-D_2\sqrt{(\alpha-1)(1+x)}\right]\cosh\sqrt{(\alpha-1)(1+x)} \qquad (27)
\end{aligned}
$$

where D_1 and D_2 are new arbitrary constants. This is the third category of charged solutions found by Hansraj and Maharaj.

The exact solutions (25), (26) and (27) were comprehensively studied by Hansraj and Maharaj [19], and it was shown that these solutions correspond to a charged relativistic sphere which is realistic as all conditions for physically acceptability are met. The condition of causality is satisfied and the speed of light is greater than the speed of sound.

6.2 Case II: Maharaj and Komathiraj charged stars

If $b=1$, then (23) becomes

$$
\begin{aligned}
y &= d_1(1+ax)^{\frac{1}{2}}\left[1-(n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{1-a}\right)^i\frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+x)^i\right] \\
&\quad +d_2\,(1+x)^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{1-a}\right)^i\frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}(1+x)^i\right] \qquad (28)
\end{aligned}
$$

for $a-1+\alpha=a(2n+1)(2n+3)$. When $b=1$ then (24) gives

$$
\begin{aligned}
y &= d_3(1+ax)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{1-a}\right)^i\frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}(1+x)^i\right] \\
&\quad +d_4\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{1-a}\right)^i\frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+x)^i\right] \qquad (29)
\end{aligned}
$$

for $a-1+\alpha=4an(n-1)$. The two categories of solutions (28) and (29) correspond to the Maharaj and Komathiraj [17] model for a compact sphere in electric fields. The Maharaj and Komathiraj charged stars have a simple form in terms of elementary functions; they are physically reasonable and contain the Durgapal and Bannerji [16] model and other exact models corresponding to neutron stars.

6.3 Case III: Finch and Skea neutron stars

When $\alpha=0$, we obtain

$$y=\left[D_1-D_2\sqrt{1+x}\right]\cos\sqrt{1+x}+\left[D_2+D_1\sqrt{1+x}\right]\sin\sqrt{1+x} \tag{30}$$

from (25). Thus, we regain the Finch and Skea [20] model for a neutron star when the electromagnetic field is absent. The Finch and Skea neutron star model has been shown to satisfy all the physical criteria for an isolated spherically symmetric stellar uncharged source. It is for this reason that this model has been used by many researchers to model the interior of neutron stars.

6.4 Case IV: Durgapal and Bannerji neutron stars

If we take $\alpha=0$ and $n=0$ then $2a+b=0$, and we get

$$y=d_1(1+ax)^{\frac{1}{2}}(5-4ax)+d_2(1-2ax)^{\frac{3}{2}}$$

from (23). If we set $a=-\frac{1}{2}$ (*ie.* $b=1$), then it is easy to verify that this equation becomes

$$y=c_1(2-x)^{\frac{1}{2}}(5+2x)+c_2(1+x)^{\frac{3}{2}} \tag{31}$$

where $c_1=d_1/3\sqrt{2}$ and $c_2=d_2$ are new arbitrary constants. Thus we have regained the Durgapal and Bannerji [16] neutron star model. This model satisfies all physical criteria for acceptability and has been utilised by many researchers to model uncharged neutron stars.

6.5 Case V: Tikekar Superdense Stars

If we take $\alpha=0$ and $n=2$ then $7a+b=0$, and we find

$$y=d_3(1+ax)(1-7ax)^{\frac{3}{2}}+d_4\left[1+\frac{1}{2}(1-7ax)-\frac{1}{8}(1-7ax)^2\right]$$

from (24). If we set $a=-1$ (*ie.* $b=7$) and let $\tilde{x}=\sqrt{1-x}$ then this equation becomes

$$y=c_1\tilde{x}(1-\frac{7}{8}\tilde{x}^2)^{\frac{3}{2}}+c_2\left[1-\frac{7}{2}\tilde{x}^2+\frac{49}{24}\tilde{x}^4\right] \tag{32}$$

where $c_1 = d_3 8^{\frac{3}{2}}$ and $c_2 = -d_4/3$ are new arbitrary constants. Thus we have regained the Tikekar [21] model for superdense neutron star from our general solution. The Tikekar superdense model plays an important role in describing highly dense matter, cold compact matter and core-envelope models for relativistic stars. The Tikekar relativistic star falls into a more general class of models with spheroidal spatial geometry found by Maharaj and Leach [12]; this class can be generalised to include the presence of an electric field as shown by Komathiraj and Maharaj [22].

7 Physical Analysis

In this section we demonstrate that the exact solutions found in this paper are physically reasonable and may be used to model a charged relativistic sphere. We observe from (22a) and (22b) that the gravitational potentials $e^{2\nu}$ and $e^{2\lambda}$ are continuous in the stellar interior and nonzero at the centre for all values of the parameters a, b and α. From (22c), we can express the variable x in terms of the energy density ρ only as

$$x = \frac{1}{4b}\left\{C[2(b-a)-\alpha]\rho^{-1} \pm \sqrt{C^2[2(b-a)-\alpha]^2\rho^{-2} + 8C[4(b-a)+\alpha]\rho^{-1} - 4}\right\}.$$

Therefore from (22d), the isotropic pressure p can be express in terms of ρ only. Thus all the forms of the solutions presented in this paper satisfy the barotropic equation of state $p = p(\rho)$ which is a desirable feature. Note that many of the solutions appeared in the literature do not satisfy this property.

To illustrate the graphical behaviour of the matter variables in the stellar interior we consider the particular solution (25). In this case the line element becomes

$$\begin{aligned} ds^2 &= -A^2\left[\left(D_1 - D_2\sqrt{(1-\alpha)(1+r^2)}\right)\cos\sqrt{(1-\alpha)(1+r^2)} \right. \\ &\quad \left. + \left(D_2 + D_1\sqrt{(1-\alpha)(1+r^2)}\right)\sin\sqrt{(1-\alpha)(1+r^2)}\right]^2 dt^2 \\ &\quad +(1+Cr^2)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \tag{33}$$

For simplicity we make the choice $A = 1, C = 1, D_1 = 1, D_2 = 4$ in the metric (33). We choose $\alpha = \frac{1}{2}$ for the charged and we consider the interval $0 \leq r \leq 1$ to generate the relevant plots.

We utilised the software package Mathematica to generate the plots for ρ, p, E^2 and $\frac{dp}{d\rho}$ respectively. The dotted line corresponds to $\alpha = \frac{1}{2}$ and $E^2 \neq 0$; the solid line corresponds to $\alpha = 0$ and $E^2 = 0$. In Fig.1 we have plotted the energy density on the interval $0 \leq r \leq 1$. It can be easily seen that the energy densities in both cases are positive and continuous at the centre; it is a monotonically decreasing function throughout the interior of the star from centre to the boundary. In Fig.2 we have plotted the behaviour of the isotropic pressure. The pressure p remains regular in the interior and is monotonically decreasing. The role of the electromagnetic field is highlighted in figures 1 and 2: the effect of E^2 is to produce smaller values for ρ and p. From figures 1 and 2 we observe that the presence of E does not significantly affect ρ but has a much greater influence on p closer to the centre. We believe that this follows directly from our choice (10) for the electric field intensity; other choices of E generate different profiles as indicated in [18]. The electric field intensity E^2 is given in Fig.3 which is positive, continuous and monotonically increasing. In Fig.4 we have plotted $\frac{dp}{d\rho}$ on the interval $0 \leq r \leq 1$ for both charged and uncharged cases. We observe that $\frac{dp}{d\rho}$ is always positive and less than unity. This indicates that the speed of the sound is less than the speed of the light and causality is maintained. Note that the effect of the electromagnetic field is to produce lower values for $\frac{dp}{d\rho}$ and the speed of sound is decreased when $\alpha \neq 0$. Hence we have shown that the solution (25), for our particular chosen parameter values, satisfies the requirements for a physically reasonable charged body.

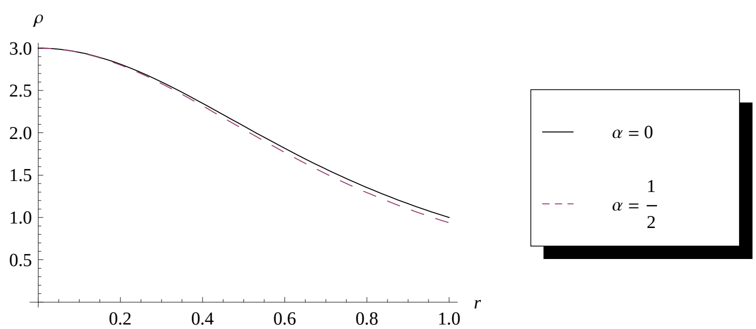

Figure 1: Energy density in geometrised units.

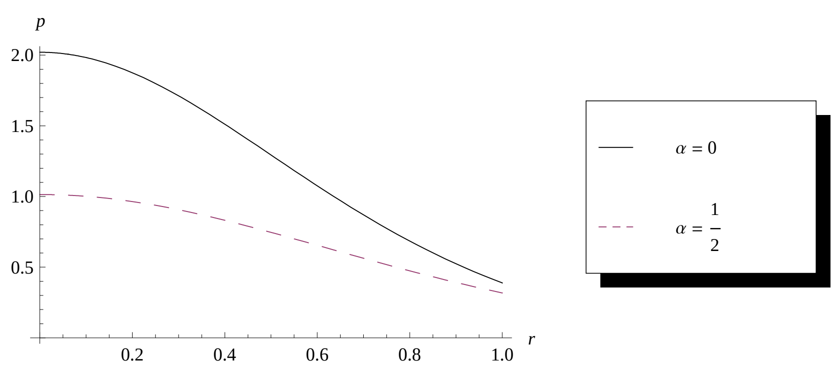

Figure 2: Matter pressure in geometrised units.

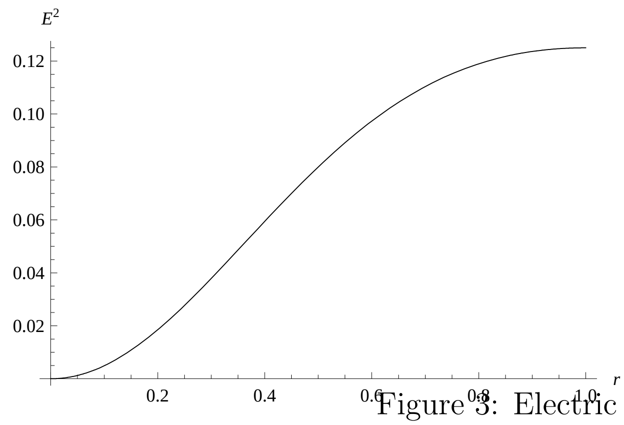

Figure 3: Electric field in geometrised units.

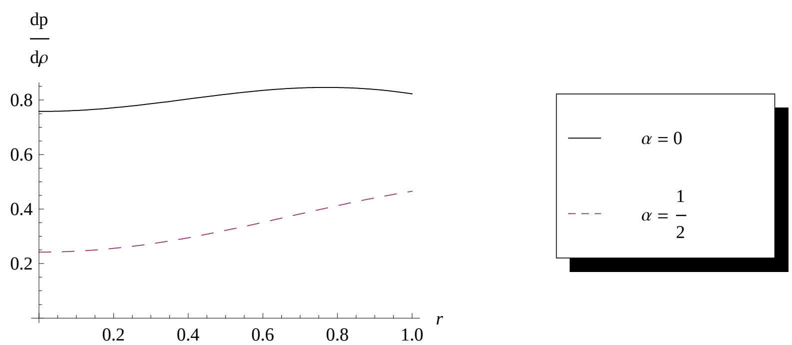

Figure 4: The gradient $dp/d\rho$ in geometrised units.

8 Discussion

In this paper we have found a new class of exact solutions (21) to the Einstein-Maxwell system of equations (5a)-(5d). We demonstrated that solutions (23) and (24) in terms of elementary functions can be extracted from the general class. If $\alpha = 0$ then we regain uncharged solutions of the Einstein field equations which may be new. The charged solutions of Hansraj and Maharaj [19] and Maharaj and Komathiraj [17] were shown to be special cases of our general class; the uncharged neutron star models of Finch and Skea [20], Durgapal and Bannerji [16] and Tikekar [21] are regained when the electromagnetic field vanishes. The simple form of the exact solutions in terms of polynomials and algebraic functions facilitates the analysis of the physical features of a charged sphere. The solutions found satisfy a barotropic equation of state. For particular parameter values we plotted the behaviour of ρ, p, E^2 and $\frac{dp}{d\rho}$ and showed that they were physically reasonable. A pleasing feature of our plots is that we can distinguish between charged and uncharged exact solutions. The presence of charge leads to smaller values of ρ, p and $\frac{dp}{d\rho}$ in the figures generated. This indicates that the presence of charge can dramatically affect the behaviour of the matter and gravitational variables.

Acknowledgements

ST thanks the National Research Foundation and the University of KwaZulu-Natal for financial support, and is grateful to Eastern University, Sri Lanka for study leave. SDM acknowledges that this work is based upon research supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation.

Appendix

In the Appendix we derive the elementary functions (23) and (24).

On substituting $r = 0$ into equation (17) we find that

$$c_{i+1} = \frac{4ai(i-1) - ((a-b)+\alpha)}{(a-b)(2i+2)(2i-1)} c_i, \quad i \geq 0. \tag{34}$$

If we set $a - b + \alpha = 4an(n-1)$, where n is a fixed integer and we assume that $a \neq 0$, then $c_{n+1} = 0$. It is easy to see that the subsequent coefficients $c_{n+2}, c_{n+3}, c_{n+4}, \ldots$ vanish and equation (34) has the solution

$$c_i = -n(n-1)\left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!} c_0 \ , 1 \leq i \leq n. \tag{35}$$

Then from equation (15) (when $r = 0$) and (35) we generate

$$Y_1 = c_0 \left[1 - n(n-1) \sum_{i=1}^{n} \left(\frac{4a}{b-a}\right)^i \frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!} X^i\right] \tag{36}$$

where $a - b + \alpha = 4an(n-1)$.

On substituting $r = \frac{3}{2}$ into (17), we get

$$c_{i+1} = \frac{a(2i+3)(2i+1) - ((a-b)+\alpha)}{(a-b)(2i+5)(2i+2)} c_i, \quad i \geq 0. \tag{37}$$

If we set $a - b + \alpha = a(2n+3)(2n+1)$, where n is a fixed integer and we assume that $a \neq 0$, then $c_{n+1} = 0$. Also we see that the subsequent coefficients $c_{n+2}, c_{n+3}, c_{n+4}, \ldots$ vanish and equation (37) is solved to give

$$c_i = \left(\frac{4a}{b-a}\right)^i \frac{3(2i+2)(n+i+1)!}{(n+1)(n-i)!(2i+3)!} c_0, \qquad 1 \leq i \leq n. \tag{38}$$

Then from equations (15) (when $r = \frac{3}{2}$) and (38) we generate

$$Y_1 = c_0 X^{\frac{3}{2}} \left[1 + \frac{3}{(n+1)} \sum_{i=1}^{n} \left(\frac{4a}{b-a} \right)^i \frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!} X^i \right] \tag{39}$$

where $a - b + \alpha = a(2n+3)(2n+1)$. The elementary functions (36) and (39) comprise the first solution of the differential equation (11) for appropriate values of $a - b + \alpha$.

We take the second solution of (11) to be of the form

$$Y_2 = [aX - (a-b)]^{\frac{1}{2}} \, u(X)$$

where $u(X)$ is an arbitrary function. On substituting Y_2 into (11) we obtain

$$4X \left[aX - (a-b) \right] \ddot{u} - 2 \left[2aX + (a-b) \right] \dot{u} - \left[2a - b + \alpha \right] u = 0 \tag{40}$$

where dots denote differentiation with respect to X. We write the solution of the differential equation (40) in the series form

$$u = \sum_{n=0}^{\infty} c_n X^{n+r} \, , \qquad c_0 \neq 0. \tag{41}$$

On substituting (41) into the differential equation (40) we find

$$\begin{aligned} 2(a-b)c_0 r[-2(r-1)+1]X^{r-1} - \sum_{n=0}^{\infty} \big(2(a-b)c_{n+1}(n+1+r)[2(n+r)-1] & \\ -c_n[4a(n+r)^2 - (2a-b+\alpha)] \big) X^{n+r} &= 0. \end{aligned} \tag{42}$$

Setting the coefficient of X^{r-1} in (42) to zero we find

$$(a-b)c_0 r[2(r-1)-1] = 0.$$

which is the indicial equation. Since $c_0 \neq 0$ and $a \neq b$ we must have $r = 0$ or $r = \frac{3}{2}$. Equating the coefficient of X^{n+r} in (42) to zero we find that

$$c_{n+1} = \frac{4a(n+r)^2 - (2a-b+\alpha)}{2(a-b)(n+r+1)[2(n+r)-1]} c_n \tag{43}$$

We establish a general structure for the coefficients by considering the leading terms.

On substituting $r = 0$ in equation (43) we obtain

$$c_{i+1} = \frac{4ai^2 - (2a-b+\alpha)}{(a-b)(2i+2)(2i-1)} c_i. \tag{44}$$

We assume that $a - b + \alpha = a(2n+3)(2n+1)$ where n is a fixed integer. Then $c_{n+2} = 0$ from (44). Consequently the remaining coefficients $c_{n+3}, c_{n+4}, c_{n+5}, \ldots$ vanish and equation (44) has the solution

$$c_i = -(n+1) \left(\frac{4a}{b-a} \right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!} c_0, \quad 1 \leq i \leq n+1. \tag{45}$$

Then from the equations (41) (when $r = 0$) and (45) we find

$$u = c_0 \left[1 - (n+1) \sum_{i=1}^{n+1} \left(\frac{4a}{b-a} \right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!} X^i \right].$$

Hence we generate the result

$$Y_2 = c_0 \left[aX - (a-b) \right]^{\frac{1}{2}} \left[1 - (n+1) \sum_{i=1}^{n+1} \left(\frac{4a}{b-a} \right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!} X^i \right] \tag{46}$$

where $a - b + \alpha = a(2n+3)(2n+1)$.

On substituting $r = \frac{3}{2}$ into equation (43) we obtain

$$c_{i+1} = \frac{a(2i+3)^2 - (2a - b + \alpha)}{(a-b)(2i+5)(2i+2)} c_i. \tag{47}$$

We assume that $a - b + \alpha = 4an(n-1)$ where n is a fixed integer. Then $c_{n-1} = 0$ from (47). Consequently the remaining coefficients $c_n, c_{n+1}, c_{n+2}, ...$ vanish and (47) can be solved to obtain

$$c_i = \left(\frac{4a}{b-a} \right)^i \frac{3(2i+2)(n+i)!}{n(n-1)(2i+3)!(n-i-2)!} c_0, \quad i \leq n-2. \tag{48}$$

Then from the equations (41) (when $r = \frac{3}{2}$) and (48) we have

$$u = c_0 X^{\frac{3}{2}} \left[1 + \frac{3}{n(n-1)} \sum_{i=1}^{n-2} \left(\frac{4a}{b-a} \right)^i \frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!} X^i \right].$$

Hence we generate the result

$$Y_2 = c_0 \left[aX - (a-b) \right]^{\frac{1}{2}} X^{\frac{3}{2}} \left[1 + \frac{3}{n(n-1)} \sum_{i=1}^{n-2} \left(\frac{4a}{b-a} \right)^i \frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!} X^i \right] \tag{49}$$

where $a - b + \alpha = 4an(n-1)$. The functions (46) and (49) generate the second solution of the differential equation (11).

The solutions found can be written in terms of two classes of elementary functions. We have the first category of solutions

$$\begin{aligned} Y \quad = \quad & D_1 \left[aX - (a-b) \right]^{\frac{1}{2}} \left[1 - (n+1) \sum_{i=1}^{n+1} \left(\frac{4a}{b-a} \right)^i \frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!} X^i \right] \\ & + D_2 X^{\frac{3}{2}} \left[1 + \frac{3}{(n+1)} \sum_{i=1}^{n} \left(\frac{4a}{b-a} \right)^i \frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!} X^i \right] \end{aligned} \tag{50}$$

for $a-b+\alpha=a(2n+3)(2n+1)$, where D_1 and D_2 are arbitrary constants. In terms of x the solution (50) becomes

$$\begin{aligned} y = \quad & d_1(1+ax)^{\frac{1}{2}}\left[1-(n+1)\sum_{i=1}^{n+1}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i)!}{(2i)!(n-i+1)!}(1+bx)^i\right] \\ +d_2 \quad & (1+bx)^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}(1+bx)^i\right] \end{aligned} \tag{51}$$

where $d_1=D_1\sqrt{b}$ and $d_2=D_2$ are new arbitrary constants. The second category of solutions is given by

$$\begin{aligned} Y \quad = \quad & D_1\left[aX-(a-b))\right]^{\frac{1}{2}}X^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}X^i\right] \\ & +D_2\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}X^i\right] \end{aligned} \tag{52}$$

for $a-b+\alpha=4an(n-1)$, where D_1 and D_2 are arbitrary constants. In terms of x the solution (52) becomes

$$\begin{aligned} y \quad = \quad & d_3(1+ax)^{\frac{1}{2}}(1+bx)^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\left(\frac{4a}{b-a}\right)^i\frac{(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}(1+bx)^i\right] \\ & +d_4\left[1-n(n-1)\sum_{i=1}^{n}\left(\frac{4a}{b-a}\right)^i\frac{(2i-1)(n+i-2)!}{(2i)!(n-i)!}(1+bx)^i\right] \end{aligned} \tag{53}$$

where $d_3=D_1\sqrt{b}$ and $d_4=D_2$ are new arbitrary constants.

References

1. Ivanov BV. Static charged perfect fluid sphere in general relativity. *Physical Review D* 2002; **65**:104001.

2. Sharma R, Mukherjee S, Maharaj SD. General solution for a class of static charged spheres. *General Relativity and Gravitation* 2001; **33**:999-1009.

3. Patel LK, Koppar SS. A charged analogue of the Vaidya-Tikekar solution. *Australian Journal of Physics* 1987; **40**:441-447.

4. Tikekar R, Singh GP. Interior Reissner-Nordstrom metric on spheroidal space-times. *Gravitation and Cosmology* 1998; **4**:294-296.

5. Mukherjee B. Static spherically symmetric perfect fluid distribution in presence of electromagnetic field. *Acta Physica Hungarica* 2001; **13**:243-252.

6. Gupta YK, Kumar M. A superdense star model as charged analogue of Schwarzschild's interior solution. *General Relativity and Gravitation* 2005; **37**:575-583.

7. Thomas VO, Ratanpal BS, Vinodkumar P C. Core-envelope models of superdense star with anisotropic envelope. *International Journal of modern Physics D* 2005; **14**:85-96.

8. Tikekar R, Thomas VO. Relativistic fluid sphere on pseudo-spheroidal space-time. *Pramana Journal of Physics* 1998; **50**:95-103.

9. Paul BC, Tikekar R. A core-envelope model of compact stars. *Gravitation and Cosmology* 2005; **11**:244-248.

10. Mak MK, Harko T. Quark stars admitting a one-parameter group of conformal motion. *International Journal of Modern Physics D* 2004; **13**: 149-156.

11. Komathiraj K, Maharaj SD. Analytical models for quark stars. *International Journal of Modern Physics D* 2007; **16**:1803-1811.

12. Maharaj SD, Leach PGL. Exact solutions for the Tikekar superdense star. *Journal of Mathematical Physics* 1996; **37**:430-437.

13. Thirukkanesh S, Maharaj SD. Exact models for isotropic matter. *Classical and Quantum Gravity* 2006; **23**:2697-2709.

14. Maharaj SD, Thirukkanesh S. Generating potentials via difference equations. *Mathematical Methods in the Applied Sciences* 2006; **29**:1943-1952.

15. John AJ, Maharaj SD. An exact isotropic solution. *Il Nuovo Cimento B* 2006; **121**:27-33.

16. Durgapal MC, Bannerji R. New analytical stellar model in general relativity. *Physical Review D* 1983; **27**:328-331.

17. Maharaj SD, Komathiraj K. Generalized compact spheres in electric field. *Classical and Quantum Gravity* 2007; **24**:4513-4524.

18. Komathiraj K, Maharaj SD. Classes of exact Einstein-Maxwell solutions. *General Relativity and Gravitation* 2007; **39**:2079-2093.

19. Hansraj S, Maharaj SD. Charged analogue of Finch-Skea stars. *International Journal of Modern Physics D* 2006; **15**:1311-1327.

20. Finch MR, Skea JEF. A realistic stellar model based on an ansatz of Duorah and Ray. *Classical and Quantum Gravity* 1989; **6**:467-476.

21. Tikekar R. Exact model for a relativistic star. *Journal of Mathematical Physics* 1990; **31**:2454-2458.

22. Komathiraj K Maharaj SD. Tikekar superdense stars in electric fields. *Journal of Mathematical Physics* 2007; **48**:042501.