Exact models for isotropic matter

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Abstract. We study the Einstein-Maxwell system of equations in spherically symmetric gravitational fields for static interior spacetimes. The condition for pressure isotropy is reduced to a recurrence equation with variable, rational coefficients. We demonstrate that this difference equation can be solved in general using mathematical induction. Consequently we can find an explicit exact solution to the Einstein-Maxwell field equations. The metric functions, energy density, pressure and the electric field intensity can be found explicitly. Our result contains models found previously including the neutron star model of Durgapal and Bannerji. By placing restrictions on parameters arising in the general series we show that the series terminate and there exist two linearly independent solutions. Consequently it is possible to find exact solutions in terms of elementary functions, namely polynomials and algebraic functions.

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1. Introduction

The first spacetime studied, modelling the interior of the relativistic sphere with uniform energy density, is the Schwarzschild interior solution which was found about ninety years ago. Since then considerable time and energy has been spent in finding exact solutions to the Einstein field equations for the interior spacetime that matches to the Schwarzschild exterior. The principal reason for this activity is that these solutions to the field equations for spherically symmetric gravitational fields are necessary in the description of compact objects in relativistic astrophysics. The models generated are used to describe relativistic spheres with strong gravitational fields as is the case in neutron stars. The detailed treatments of Stephani *et al* [1] and Delgaty and Lake [2] for static, spherically symmetric models provide a comprehensive collection of interior spacetimes, satisfying a variety of criteria for physical admissability, that match to the Schwarzschild exterior spacetime. It is important to note that only a few of these solutions correspond to nonsingular gravitational potentials with a physically acceptable energy momentum tensor and a barotropic equation of state. A sample of the exact solutions to the field equations, which satisfy all the physical requirements for a relativistic star, is contained in the models of the Durgapal and Bannerji [3], Durgapal and Fuloria [4], Finch and Skea [5], Ivanov [6], Lake [7], Maharaj and Leach [8] and Sharma and Mukherjee [9], amongst others.

In the past most of the solutions found have been obtained in an *ad hoc* manner. We expect that a more systematic and formal study, such as the treatment of Maharaj and Chaisi [10] for anisotropic matter, should lead to new classes of solution. Clearly there is a need to systematically study the mathematical properties and features of the underlying nonlinear differential equations. The analysis of John [11] indicates that reducing the condition of pressure isotropy to a recurrence relation with real, rational coefficients leads to new mathematical and physical insights in the Einstein equations for neutral matter. We attempt to perform similar analysis here in the coupled Einstein-Maxwell equations for charged matter. In this more general case we find that the condition of pressure isotropy leads to a new recurrence relation which can be solved in general. The Einstein-Maxwell system is important in the description of a relativistic star in the presence of an electromagnetic field. It is interesting to observe that, in the presence of charge, the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided. In this situation the gravitational attraction is counterbalanced by the repulsive Coulombian force in addition to the pressure gradient. Consequently the Einstein-Maxwell system, for a charged star, has attracted considerable attention in various physical investigations including Mukherjee [12] and Sharma *et al* [13].

In this paper we seek new exact solutions to the Einstein field equations, using a systematic series analysis, which may be used to describe the interior gravitational profile of a relativistic sphere. The approach produces a number of difference equations which we demonstrate can be solved from first principles. In section 2 we first express the Einstein equations for neutral matter and the Einstein-Maxwell system for charged matter as equivalent sets of differential equations utilising a transformation due to Durgapal and Bannerji [3]. We choose particular forms for one of the gravitational potentials and the the electric field intensity, which we believe has not been studied before. This enables us to simplify the condition of pressure isotropy in section 3 to a second order linear equation in the remaining gravitational potential. We assume a series form for this function which yields a difference equation which we manage to solve using mathematical induction. It is then possible to exhibit a new exact solution to the Einstein-Maxwell field equations which can be written explicitly as shown in section 4. We consider two particular cases in section 5 which contain exact solutions found previously. In section 6 we demonstrate that it is possible to find two linearly independent solutions to the condition of pressure isotropy in terms of elementary functions by placing restrictions on parameters, that appear in the general solution, thereby permitting the series to terminate. In section 7 we express the general solution of the Einstein Maxwell system in terms of polynomials and algebraic functions. We briefly discuss some of the physical properties of our solutions in section 8.

2. Field equations

On physical grounds the gravitational field should be static and spherically symmetric for describing the internal structure of a dense compact relativistic sphere. Consequently we can find coordinates (t, r, θ, ϕ) such that the line element is of the form

$$\mathrm{d}s^2 = -e^{2\nu(r)}\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{1}$$

For neutral perfect fluids the Einstein field equations can be expressed as follows

$$\frac{1}{r^2}[r(1-e^{-2\lambda})]' = \rho \tag{2a}$$

$$-\frac{1}{r^2}\,(1-e^{-2\lambda}) + \frac{2\nu'}{r}\,e^{-2\lambda} = p \tag{2b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p \tag{2c}$$

for the spherically symmetric line element (1). The energy density ρ and the pressure p are measured relative to the comoving fluid 4-velocity $u^a = e^{-\nu}\delta^a_0$ and primes denote differentiation with respect to the radial coordinate r. In the field equations $(2a)$-$(2c)$ we are using units where the coupling constant $\frac{8\pi G}{c^4} = 1$ and the speed of light $c = 1$. The system of equations $(2a)$-$(2c)$ governs the behaviour of the gravitational field for a neutral perfect fluid. A different but equivalent form of the field equations is obtained if we define a new independent variable x, and new functions y and Z, as follows

$$A^2y^2(x) = e^{2\nu(r)}, \;\; Z(x) = e^{-2\lambda(r)}, \;\; x = Cr^2 \tag{3}$$

so that the line element (1) becomes

$$\mathrm{d}s^2 = -A^2y^2\mathrm{d}t^2 + \frac{1}{4CxZ}\mathrm{d}x^2 + \frac{x}{C}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2).$$

In the equations (3) the quantities A and C are arbitrary constants. Under the transformation (3) the system $(2a)$-$(2c)$ has the form

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} \tag{4a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} \tag{4b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y = 0 \tag{4c}$$

where the dots denotes differentiation with respect to the variable x. The set $(4a)$-$(4c)$ is a system of three equations in the four unknowns ρ, p, y and Z.

A generalisation of the system $(2a)$-$(2c)$ is the Einstein-Maxwell field equations given by

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho + \frac{1}{2}E^2 \tag{5a}$$

$$-\frac{1}{r^2}\left(1-e^{-2\lambda}\right) + \frac{2\nu'}{r}e^{-2\lambda} = p - \frac{1}{2}E^2 \tag{5b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p + \frac{1}{2}E^2 \tag{5c}$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)'. \tag{5d}$$

where E is the electric field intensity and σ is the charge density. When the electric field $E = 0$ then the Einstein-Maxwell equations $(5a)$-$(5d)$ reduce to the Einstein equations $(2a)$-$(2c)$ for neutral matter. The system of equations $(5a)$-$(5d)$ governs the behaviour of the gravitational field for a charged perfect fluid. If we use the transformation (3) then the Einstein-Maxwell system $(5a)$-$(5d)$ becomes

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{C} + \frac{E^2}{2C} \tag{6a}$$

$$4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p}{C} - \frac{E^2}{2C} \tag{6b}$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2x}{C}\right)y = 0 \tag{6c}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2 \tag{6d}$$

which may be easier to integrate in certain situations.

3. Specifying Z and E

We examine a particular form of the Einstein-Maxwell field equations $(6a)$-$(6d)$ by making explicit choices for the gravitational potential Z and the electric field intensity E. The system $(6a)$-$(6d)$ comprises four equations in six unknowns Z, y, ρ, p, E and σ. By specifying the gravitational potential Z and electric field intensity E we are in a position to integrate the condition of pressure isotropy ($6c$). The solution of the Einstein-Maxwell system then follow. We make the choice

$$Z = \frac{1+kx}{1+x} \tag{7}$$

where k is a real constant. In (7) we take $k \neq 1$. If $k = 1$ then the metric function $e^{2\lambda}$=1 and the energy density is $\rho = -\frac{E^2}{2}$. To avoid negative energy densities, which are not physical for barotropic stars, we consequently take $k \neq 1$. The choice (7) was also made by Maharaj and Mkhwanazi [14] and in their analysis of uncharged stars. Our objective is to confirm that this type of potential is also consistent with nonvanishing electromagnetic fields. Note that our choice contains, as a special case, the Durgapal and Bannerji [3] solution, which is widely applied as a relativistic model for neutral stars. Only the solutions for the cases $k = \frac{1}{2}$ and $k = -\frac{1}{2}$ were documented previously for the uncharged case when $E = 0$. Other physically reasonable choices of the gravitational potential Z are possible; we have chosen the form (7) as it produces charged and uncharged solutions which are necessary for a realistic model.

Upon substituting (7) in equation ($6c$) we obtain

$$4(1+kx)(1+x)\ddot{y} + 2(k-1)\dot{y} + \left(1 - k - \frac{E^2(1+x)^2}{Cx}\right) y = 0. \tag{8}$$

As the differential equation (8) is difficult to solve we first introduce the transformation

$$1+x = KX \tag{9a}$$

$$K = \frac{k-1}{k} \tag{9b}$$

$$Y(X) = y(x) \tag{9c}$$

so as to obtain a more convenient form. Substituting ($9a$)-($9c$) in the differential equation (8) we obtain

$$4X(1-X)\frac{d^2Y}{dX^2} - 2\frac{dY}{dX} + \left(K + \frac{K^2(1-K)E^2X^2}{C(KX-1)}\right) Y = 0 \tag{10}$$

in terms of the new dependent and independent variables Y and X respectively. The differential equation (10) may be integrated once the electric field E is specified. A variety of choices for E is possible but only a few are physically reasonable and generate solutions in closed form. We observe that the particular choice

$$E^2 = \frac{\alpha C}{K^2(1-K)} \frac{KX-1}{X^2} \tag{11}$$

where α is a constant, simplifies (10). The electric field defined in (11) vanishes at the centre of the star, and remains continuous and bounded in the interior of the star for a wide range of values of the parameter k. Thus the choice for E is physically reasonable and it is a useful form to study the gravitational behaviour of charged stars. With the help of (11) we find that (10) takes the simpler form

$$4X(1-X)\frac{d^2Y}{dX^2} - 2\frac{dY}{dX} + (K+\alpha)Y = 0. \tag{12}$$

This is a special case of the hypergeometric equation. When $\alpha = 0$ the differential equation (12) becomes

$$4X(1-X)\frac{d^2Y}{dX^2} - 2\frac{dY}{dX} + KY = 0 \tag{13}$$

and there is no charge.

4. General series solution

Since (12) is the hypergeometric equation it is not possible to express the general solution in terms of elementary functions for all $K+\alpha$. In general the solution will be given in terms of special functions. The representation of the solution in a simple form is necessary for a detailed physical analysis. Hence we attempt to obtain a general solution to the differential equation (12) in series form. Later we show that it is possible to extract solutions in terms of algebraic functions and polynomials.

Since $X = 0$ is a regular singular point of the differential equation (12) we can apply the method of Frobenius about $X = 0$. We assume that the solution of the differential equation (12) is of the form

$$Y = \sum_{n=0}^{\infty} c_n X^{n+r}, \quad c_0 \neq 0 \tag{14}$$

where c_n are the coefficients of the series and r is a constant. For a legitimate solution we need to determine the coefficients c_n as well as the parameter r. Substituting (14) in the differential equation (12) we obtain

$$2c_0 r[2(r-1)-1]X^{r-1} +$$
$$\sum_{n=0}^{\infty} \left(2c_{n+1}(n+1+r)[2(n+r)-1] - c_n[4(n+r)(n+r-1)-(K+\alpha)]\right) X^{n+r} = 0. \tag{15}$$

The coefficients of the various powers of X must vanish. Equating the coefficient of X^{r-1} in (15) to zero we obtain the indicial equation

$$2c_0 r[2(r-1)-1] = 0.$$

Since $c_0 \neq 0$ we must have $r = 0$ or $r = \frac{3}{2}$. Equating the coefficient of X^{n+r} in (15) to zero we obtain

$$c_{n+1} = \frac{4(n+r)(n+r-1)-(K+\alpha)}{2(n+1+r)[2(n+r)-1]} c_n, \quad n \geq 0 \tag{16}$$

which is the fundamental difference equation governing the structure of the solution.

We can establish a general structure for the coefficients by considering the leading terms. The coefficients $c_1, c_2, c_3, \ldots$can all be written in terms of the leading coefficient c_0 and we generate the expression

$$c_{n+1} = \prod_{p=0}^{n} \frac{4(p+r)(p+r-1)-(K+\alpha)}{2(p+1+r)[2(p+r)-1]} c_0 \tag{17}$$

where the symbol $\prod$ denotes multiplication. It is possible to establish that the result (17) holds for all nonnegative integers using the principle of mathematical induction.

We can now generate two linearly independent solutions, y_1 and y_2, from (14) and (17). For the parameter value $r = 0$ we obtain the first solution

$$Y_1 = c_0\left[1 + \sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{4p(p-1)-(K+\alpha)}{2(p+1)(2p-1)}X^{n+1}\right]$$

$$y_1 = c_0\left[1 + \sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{4p(p-1)-(K+\alpha)}{2(p+1)(2p-1)}\left(\frac{1+x}{K}\right)^{n+1}\right]. \tag{18}$$

For the parameter value $r = \frac{3}{2}$ we obtain the second solution

$$Y_2 = c_0X^{\frac{3}{2}}\left[1 + \sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{(2p+3)(2p+1)-(K+\alpha)}{(2p+5)(2p+2)}X^{n+1}\right]$$

$$y_2 = c_0\left(\frac{1+x}{K}\right)^{\frac{3}{2}}\left[1 + \sum_{n=0}^{\infty}\prod_{p=0}^{n}\frac{(2p+3)(2p+1)-(K+\alpha)}{(2p+5)(2p+2)}\left(\frac{1+x}{K}\right)^{n+1}\right]. \tag{19}$$

Thus the general solution to the differential equation (8), for the choice (11), is given by

$$y = ay_1(x) + by_2(x) \tag{20}$$

where a and b are arbitrary constants, $K = \frac{k-1}{k}$ and y_1 and y_2 are given by (18) and (19) respectively. By inspection it is clear that y_1 and y_2 are linearly independent functions. From (6a)-(6d), (18) and (19) the general solution to the Einstein-Maxwell system becomes

$$e^{2\lambda} = \frac{1+x}{1+kx} \tag{21a}$$

$$e^{2\nu} = A^2y^2 \tag{21b}$$

$$\frac{\rho}{C} = \frac{(1-k)(3+x)}{(1+x)^2} - \frac{\alpha kx}{2(1+x)^2} \tag{21c}$$

$$\frac{p}{C} = 4\frac{(1+kx)}{(1+x)}\frac{\dot{y}}{y} + \frac{(k-1)}{(1+x)} + \frac{\alpha kx}{2(1+x)^2} \tag{21d}$$

$$\frac{E^2}{C} = \frac{\alpha kx}{(1+x)^2} \tag{21e}$$

where $y = ay_1(x)+by_2(x)$. We believe that (21a)-(21e) is a new solution to the Einstein-Maxwell field equations.

5. Particular cases

From the Einstein-Maxwell solution (21a)-(21d) we can generate a number of physically reasonable charged and uncharged models for particular choices of k and α. If we set $\alpha = 0$ then

$$e^{2\lambda} = \frac{1+x}{1+kx} \tag{22a}$$

$$e^{2\nu} = A^2y^2 \tag{22b}$$

$$\frac{\rho}{C} = \frac{(1-k)(3+x)}{(1+x)^2} \tag{22c}$$

$$\frac{p}{C} = 4\frac{(1+kx)}{(1+x)}\frac{\dot{y}}{y} + \frac{(k-1)}{(1+x)} \tag{22d}$$

which corresponds to a neutral relativistic star. We believe that the uncharged solution (22a)-(22d) is also a new solution to the Einstein field equations (4a)-(4c). It does not appear in the comprehensive list of solutions presented by Delgaty and Lake [2]. In the solutions (21a)-(21e) and (22a)-(22d) the gravitational potentials λ and ν are well behaved. Clearly the energy density ρ is positive at the origin if we choose $k < 1$. The pressure is finite at the origin. These are desirable features in a stellar model.

When $K + \alpha = 3$ the series in (19) terminates. It is then possible to write the exact solution to the Einstein-Maxwell system in terms of elementary functions. The explicit form of the solution is given by

$$e^{2\lambda} = \frac{(K-1)(1+x)}{(K-1-x)} \tag{23a}$$

$$e^{2\nu} = A^2\left[c_1(1+x)^{\frac{3}{2}} + c_2(K-1-x)^{\frac{1}{2}}(K+2+2x)\right]^2 \tag{23b}$$

$$\frac{\rho}{C} = \frac{2K(3+x)+(3-K)x}{2(K-1)(1+x)^2} \tag{23c}$$

$$\begin{aligned}\frac{p}{C} &= \frac{1}{K-1}\times \\ &\quad \frac{c_1(1+x)^{\frac{1}{2}}[5K-6-(K+6)x] + c_2(K-1-x)^{\frac{1}{2}}[4K-12-K^2-(12+2K)x]}{c_1(1+x)^{\frac{5}{2}} + c_2(1+x)(K-1-x)^{\frac{1}{2}}(K+2+2x)} \\ &\quad + \frac{(3-K)x}{2(1-K)(1+x)^2}\end{aligned} \tag{23d}$$

$$\frac{E^2}{C} = \frac{(3-K)x}{(1-K)(1+x)^2} \tag{23e}$$

where $K = (k-1)/k$ and $x = Cr^2$. Clearly (23a)-(23e) is a special case of the general solution (21a)-(21e). The solution (23a)-(23e) has the advantage of being given completely in terms of elementary functions which makes an analysis of the physical features of the model possible. When $K = 3$ (i.e. $k = -\frac{1}{2}$) and $\alpha = 0$ we obtain

$$e^{2\lambda} = \frac{2(1+x)}{(2-x)} \tag{24a}$$

$$e^{2\nu} = A^2\left[c_1(1+x)^{\frac{3}{2}} + c_2(2-x)^{\frac{1}{2}}(5+2x)\right]^2 \tag{24b}$$

$$\frac{\rho}{C} = \frac{3(3+x)}{2(1+x)^2} \tag{24c}$$

$$\frac{p}{C} = \frac{9}{2}\left[\frac{c_1(1+x)^{\frac{1}{2}}(1-x) - c_2(2-x)^{\frac{1}{2}}(1+2x)}{c_1(1+x)^{\frac{5}{2}} + c_2(1+x)(2-x)^{\frac{1}{2}}(5+2x)}\right] \tag{24d}$$

for an uncharged relativistic stellar model. The special case (24a)-(24d) is the same as the result of Durgapal and Bannerji [3] and Maharaj and Mhkwanazi [14]. We point out that Maharaj and Mhkwanazi [14] had a numerical mistake in their calculation of the pressure p which has been corrected in our solution. We believe that the solution (23a)-(23d) is important in the study of charged stars as it contains the Durgapal and Bannerji [3] model which has been shown to be consistent with a realistic dense star. Extensive studies of the Durgapal and Bannerji solution, as indicated in the compendium by Delgaty and Lake [2], has proved that all the criteria for physical acceptability are satisfied. It is consequently used in many astrophysical studies that model neutron stars.

6. Terminating series

The general solution (20) can be expressed in terms of the special functions, namely hypergeometric functions. For particular values of K and α the series solution can be given in terms of elementary functions as demonstrated in section 5. This is possible in general because the series (18) and (19) terminate for restricted values of the parameters K and α. Using this feature we obtain two sets of general solutions in terms of elementary functions, by determining the specific restriction on $K+\alpha$ for a terminating series, as demonstrated in the following sections.

6.1. The first solution

On substituting $r=0$ in equation (16) we obtain

$$c_{i+1} = \frac{4i(i-1)-(K+\alpha)}{(2i+2)(2i-1)}c_i, \quad i \geq 0. \tag{25}$$

If we set $K+\alpha = 4n(n-1)$, where n is a fixed integer, then $c_{n+1}=0$. Clearly the subsequent coefficients $c_{n+2}, c_{n+3}, c_{n+4}, \ldots$ vanish and equation (25) has the solution

$$c_i = 4n(n-1)\frac{(-4)^{i-1}(2i-1)(n+i-2)!}{(2i)!(n-i)!}c_0, \quad 1 \leq i \leq n. \tag{26}$$

Then from the equations (14) (when $r=0$) and (26) we obtain

$$Y_1 = c_0\left[1 + 4n(n-1)\sum_{i=1}^{n}\frac{(-4)^{i-1}(2i-1)(n+i-2)!}{(2i)!(n-i)!}X^i\right] \tag{27}$$

where $K+\alpha = 4n(n-1)$.

On substituting $r=\frac{3}{2}$ in (16) we obtain

$$c_{i+1} = \frac{(2i+3)(2i+1)-(K+\alpha)}{(2i+5)(2i+2)}c_i, \quad i \geq 0. \tag{28}$$

If we set $K+\alpha = (2n+3)(2n+1)$, where n is a fixed integer then $c_{n+1}=0$. Also the subsequent coefficients $c_{n+2}, c_{n+3}, c_{n+4}, \ldots$ vanish and equation (28) can be solved to yield

$$c_i = \frac{3(-4)^i(2i+2)(n+i+1)!}{(n+1)(n-i)!(2i+3)!}c_0, \quad 1 \leq i \leq n. \tag{29}$$

Then from the equations (14) (when $r = \frac{3}{2}$) and (29) we obtain

$$Y_1 = c_0 X^{\frac{3}{2}} \left[1 + \frac{3}{(n+1)} \sum_{i=1}^{n} \frac{(-4)^i (2i+2)(n+i+1)!}{(n-i)!(2i+3)!} X^i \right] \tag{30}$$

where $K + \alpha = (2n+3)(2n+1)$. The polynomial and algebraic functions (27) and (30) comprise the first solution of the differential equation (12) for appropriate values of $K + \alpha$.

6.2. The second solution

We can use the form of the particular solution in section 5, expressed in terms of elementary functions, to simplify the representation of the second solution. The special solution in section 5 contains terms of the form $(1-X)^{\frac{1}{2}}(1+2X)$ which is product of $(1-X)^{\frac{1}{2}}$ and a polynomial. This suggests that the second solution in general is of the form

$$Y_2 = (1-X)^{\frac{1}{2}} u(X)$$

where $u(X)$ is an arbitrary function. We now take Y_2 to be the generic second solution of (12) and explicitly determine $u(X)$. Equation (12) gives

$$4X(1-X)\ddot{u} - 2(1+2X)\dot{u} + (1+K+\alpha)u = 0 \tag{31}$$

where dots denote differentiation with respect to X.

Since $X = 0$ is a regular singular point of the differential equation (31) we can apply the method of Frobenius. We assume that the solution is of the form

$$u = \sum_{n=0}^{\infty} c_n X^{n+r}, \quad c_0 \neq 0. \tag{32}$$

On substituting (32) in the differential equation (31) we obtain

$$2c_0 r[2(r-1)-1]X^{r-1} + \sum_{n=0}^{\infty} \Big(2c_{n+1}(n+1+r)[2(n+r)-1] - c_n[4(n+r)^2 - (1+K+\alpha)] \Big) X^{n+r} = 0. \tag{33}$$

The coefficients of the various powers of X have to vanish. Setting the coefficient of X^{r-1} in (33) to zero we obtain the indicial equation

$$2c_0 r[2(r-1)-1] = 0.$$

Since $c_0 \neq 0$ we must have $r = 0$ or $r = \frac{3}{2}$ as in section 4. Equating the coefficient of X^{n+r} in (33) to zero we obtain

$$c_{n+1} = \frac{4(n+r)^2 - (1+K+\alpha)}{2(n+r+1)[2(n+r)-1]} c_n \tag{34}$$

which is the basic difference equation governing the structure of the solution.

We establish a general structure for the coefficients by considering the leading terms. On substituting $r = 0$ in equation (34) we obtain

$$c_{i+1} = \frac{4i^2 - (1+K+\alpha)}{(2i+2)(2i-1)} c_i. \tag{35}$$

We assume that $K+\alpha = (2n+3)(2n+1)$ where n is a fixed integer. Then $c_{n+2} = 0$ from (35). Consequently the remaining coefficients $c_{n+3}, c_{n+4}, c_{n+5}, \ldots$ vanish and equation (35) has the solution

$$c_i = 4(n+1)\frac{(-4)^{i-1}(2i-1)(n+i)!}{(2i)!(n-i+1)!}c_0, \quad 1 \le i \le n+1. \tag{36}$$

Then from equation (32)(when $r = 0$) and (36) we obtain

$$u = c_0\left[1 + 4(n+1)\sum_{i=1}^{n+1}\frac{(-4)^{i-1}(2i-1)(n+i)!}{(2i)!(n-i+1)!}X^i\right].$$

Hence we have the result

$$Y_2 = c_0(1-X)^{\frac{1}{2}}\left[1 + 4(n+1)\sum_{i=1}^{n+1}\frac{(-4)^{i-1}(2i-1)(n+i)!}{(2i)!(n-i+1)!}X^i\right] \tag{37}$$

where $K+\alpha = (2n+3)(2n+1)$.

On substituting $r = \frac{3}{2}$ in equation (34) we obtain

$$c_{i+1} = \frac{(2i+3)^2 - (1+K+\alpha)}{(2i+5)(2i+2)}c_i. \tag{38}$$

We assume that $K+\alpha = 4n(n-1)$ where n is a fixed integer. Then $c_{n-1} = 0$ from (38). Consequently the remaining coefficients $c_n, c_{n+1}, c_{n+2}, \ldots$ vanish and (38) can be solved to yield

$$c_i = \frac{3(-4)^i(2i+2)(n+i)!}{n(n-1)(2i+3)!(n-i-2)!}c_0, \quad i \le n-2. \tag{39}$$

Then from the equations (32) (when $r = \frac{3}{2}$) and (39) we obtain

$$u = c_0X^{\frac{3}{2}}\left[1 + \frac{3}{n(n-1)}\sum_{i=1}^{n-2}\frac{(-4)^i(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}X^i\right].$$

Hence we have the result

$$Y_2 = c_0(1-X)^{\frac{1}{2}}X^{\frac{3}{2}}\left[1 + \frac{3}{n(n-1)}\sum_{i=1}^{n-2}\frac{(-4)^i(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}X^i\right] \tag{40}$$

where $K+\alpha = 4n(n-1)$.

The solutions (37) and (40) generate the second solution of the differential equation (12) which are clearly independent from the solutions (27) and (30). The quantities (37) and (40) are products of polynomials and algebraic functions.

7. Elementary functions

Thus we have generated general solutions to the differential equation (12) by restricting the values of $K+\alpha$ so that polynomials and product of polynomials with algebraic functions are possible as solutions. Collecting these results we have the first category of solutions

$$\begin{aligned} Y &= a(1-X)^{\frac{1}{2}}\left[1 + 4(n+1)\sum_{i=1}^{n+1}\frac{(-4)^{i-1}(2i-1)(n+i)!}{(2i)!(n-i+1)!}X^i\right] \\ &\quad + bX^{\frac{3}{2}}\left[1 + \frac{3}{(n+1)}\sum_{i=1}^{n}\frac{(-4)^i(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}X^i\right] \end{aligned} \tag{41}$$

for $K+\alpha=(2n+3)(2n+1)$ where a and b are arbitrary constants. In terms of x the solution (41) becomes

$$y = a\left(\frac{K-1-x}{K}\right)^{\frac{1}{2}}\left[1+4(n+1)\sum_{i=1}^{n+1}\frac{(-4)^{i-1}(2i-1)(n+i)!}{(2i)!(n-i+1)!}\left(\frac{1+x}{K}\right)^{i}\right]$$
$$+b\left(\frac{1+x}{K}\right)^{\frac{3}{2}}\left[1+\frac{3}{(n+1)}\sum_{i=1}^{n}\frac{(-4)^{i}(2i+2)(n+i+1)!}{(n-i)!(2i+3)!}\left(\frac{1+x}{K}\right)^{i}\right]. \quad (42)$$

The second category of solutions is given by

$$Y = a(1-X)^{\frac{1}{2}}X^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\frac{(-4)^{i}(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}X^{i}\right]$$
$$+b\left[1+4n(n-1)\sum_{i=1}^{n}\frac{(-4)^{i-1}(2i-1)(n+i-2)!}{(2i)!(n-i)!}X^{i}\right] \quad (43)$$

for $K+\alpha=4n(n-1)$ where a and b are arbitrary constants. In terms of x the solution (43) becomes

$$y = a\left(\frac{K-1-x}{K}\right)^{\frac{1}{2}}\left(\frac{1+x}{K}\right)^{\frac{3}{2}}\left[1+\frac{3}{n(n-1)}\sum_{i=1}^{n-2}\frac{(-4)^{i}(2i+2)(n+i)!}{(2i+3)!(n-i-2)!}\left(\frac{1+x}{K}\right)^{i}\right]$$
$$+b\left[1+4n(n-1)\sum_{i=1}^{n}\frac{(-4)^{i-1}(2i-1)(n+i-2)!}{(2i)!(n-i)!}\left(\frac{1+x}{K}\right)^{i}\right]. \quad (44)$$

It is remarkable that the solutions (42) and (44) are expressed completely as combinations of polynomial and algebraic functions. It is rare to find general solutions, considering the nonlinearity of the gravitational interactions, to the field equations in terms of elementary functions. We have expressed our solutions in the simplest possible form. This has the advantage of simplifying the analysis of the physical properties of the dense star. Observe that our treatment has combined both the charged and uncharged cases for a relativistic star. If we substitute $\alpha=0$ in (42) and (44) then we can obtain the solutions for the uncharged case directly. Consequently our approach has the unexpected but very desirable feature of producing an uncharged (possibly new) solution to equations $(4a)$-$(4c)$ from the charged solutions when $E=0$. We believe that the solutions obtained in this paper to the Einstein $(4a)$-$(4c)$ and Einstein Maxwell $(6a)$-$(6d)$ field equations have not been found before.

From our general class of solutions (42) and (44) it is possible to generate particular solutions found previously. If we take $K=3$ and $\alpha=0$ $(n=0)$ then it is easy to verify that the equation (42) becomes

$$y = c_1(2-x)^{\frac{1}{2}}(5+2x)+c_2(1+x)^{\frac{3}{2}}$$

where $c_1=a/9$ and $c_2=b/3^{\frac{3}{2}}$ are new arbitrary constants. Thus we have regained the Durgapal and Bannerji [3] neutron star model. Other explicit functional forms for y are obtainable which could be useful in applications for a dense star. As an example suppose that $K+\alpha=8(n=2)$ then from (44) we obtain

$$y = c_1(K-1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}+c_2\left[K^2+4K(1+x)-8(1+x)^2\right].$$

where $c_1 = a/K^2$ and $c_2 = b/K^2$ are new arbitrary constants. It is now possible to generate an exact solution to the Einstein-Maxwell system (6a)-(6d) in terms of elementary functions when $K + \alpha = 8$. This is given by

$$e^{2\lambda} = \frac{(K-1)(1+x)}{(K-1-x)} \tag{45a}$$

$$e^{2\nu} = A^2\left[c_1(K-1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}} + c_2\left(K^2+4K(1+x)-8(1+x)^2\right)\right]^2 \tag{45b}$$

$$\frac{\rho}{C} = \frac{2K(3+x)+(K-8)x}{2(K-1)(1+x)^2} \tag{45c}$$

$$\frac{p}{C} = \frac{2(K-1-x)^{\frac{1}{2}}\left[c_1(1+x)^{\frac{1}{2}}(3K-4(1+x))+8c_2(K-1-x)^{\frac{1}{2}}(K-4(1+x))\right]}{(K-1)(1+x)\left[c_1(K-1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}+c_2\left(K^2+4K(1+x)-8(1+x)^2\right)\right]} - \frac{K}{(K-1)(1+x)} + \frac{(K-8)x}{2(1-K)(1+x)^2} \tag{45d}$$

$$\frac{E^2}{C} = \frac{(K-8)x}{(1-K)(1+x)^2} \tag{45e}$$

where $K = (k-1)/k$ and $x = Cr^2$. The solution (45a)-(45e) is given in a simple form which facilitates a physical analysis.

8. Discussion

We have found new solutions (21a)-(21e) to the Einstein-Maxwell system using a systematic series analysis that produces a number of difference equations which can be solved in general. A useful feature of the approach is that we regain the Durgapal and Bannerji neutron star model [3] as a special case which suggests that our class of solutions are physically reasonable. We briefly consider some physical features of the solutions of interest.

Firstly, in the general solution (21a)-(21e), when studying models of charges spheres, we should consider only those values of k for which the energy density ρ, the pressure p and the electric field intensity E are positive. The choice of k must ensure that the gravitational potential $e^{2\lambda}$ remains positive; the remaining potential $e^{2\nu}$ is necessarily positive. Clearly a wide range of charged spheres, with nonsingular potentials and matter variables, are possible for relevant choices of k. The interior metric (1) must match to the Reissner-Nordstrom exterior spacetime

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)$$

across the boundary $r = R$. This yields the relationships

$$\begin{aligned} 1-\frac{2M}{R}+\frac{Q^2}{R^2} &= A^2\left[ay_1(CR^2)+by_2(CR^2)\right]^2 \\ \left(1-\frac{2M}{R}+\frac{Q^2}{R^2}\right)^{-1} &= \frac{1+CR^2}{1+kCR^2} \end{aligned}$$

between the constants a, b, k, A and C. This shows that continuity of the metric coefficients across the boundary of the star is easily achieved. The matching conditions at the boundary may place restrictions on the function ν and its first derivative (for uncharged matter) and the pressure may be nonzero (if there is a surface layer of charge); there are sufficient free parameters available to satisfy the necessary conditions that may arise from a particular physical model under consideration.

Secondly, we observe that our solutions may be interpreted as models for anisotropic spheres (which may be charged or uncharged) where the parameter α plays the role of the anisotropy factor. The solutions found depend smoothly on the parameter α; isotropic and uncharged solutions can be regained for $\alpha = 0$. For recent analyses of the physics of anisotropic matter see Chaisi and Maharaj [15], Dev and Gleiser [16], [17] and Maharaj and Chaisi [10].

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