EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2013/2014

FIRST SEMESTER (Sep./Oct., 2015)

AM 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

Answer all questions **Time : Three hours**

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1. (a) i. Find an equation for the plane passing through three points whose position vectors are given by $(2,-1,1)$, $(3,2,-1)$ and $(-1,3,2)$.

   ii. Find the volume of the parallelepiped whose edges are represented by $\underline{a} = 2\underline{i} - 3\underline{j} + 4\underline{k}$, $\underline{b} = \underline{i} + 2\underline{j} - \underline{k}$, $\underline{c} = 3\underline{i} - \underline{j} + 2\underline{k}$.

(b) The vectors $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ are defined in terms of vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ by

$$\underline{\alpha} = \frac{\underline{b} \wedge \underline{c}}{V}, \quad \underline{\beta} = \frac{\underline{c} \wedge \underline{a}}{V}, \quad \underline{\gamma} = \frac{\underline{a} \wedge \underline{b}}{V},$$

where $V = \underline{a} \cdot \underline{b} \wedge \underline{c} \neq 0$.

Show that $\underline{\alpha} \cdot \underline{\beta} \wedge \underline{\gamma} = \dfrac{1}{V}$. Show also that any vector $\underline{r}$ can be expressed in the form

$$\underline{r} = (\underline{r} \cdot \underline{\alpha})\underline{a} + (\underline{r} \cdot \underline{\beta})\underline{b} + (\underline{r} \cdot \underline{\gamma})\underline{c}.$$

(c) Find the curvature and the torsion for the curve $x = t - \dfrac{t^3}{3}$, $y = t^2$, $z = t + \dfrac{t^3}{3}$.

2. (a) Define the following terms:

   i. the gradient of a scalar field ϕ;

   ii. the divergence of a vector field $\underline{A}$.

   (b) Show that $\underline{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = C$, where C is a constant.

   (c) Find the directional derivative of the function $\phi = x^2yz + 2xz^2$ at $(1, -2, -1)$ in the direction $2\underline{i} - \underline{j} - 2\underline{k}$.

   (d) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$. Prove that $\nabla^2(r^n\underline{r}) = n(n+3)r^{n-2}\underline{r}$, where n is a constant.

3. (a) Define the following terms:

   i. conservative vector field;

   ii. solenoidal vector field.

   (b) Show that the vector $\underline{A} = (y^2 \cos x + z^3)\underline{i} + (2y \sin x - 4)\underline{j} + (3xz^2 + 2)\underline{k}$ is a conservative force field and need not be a solenoidal vector field. Find the scalar potential ϕ such that $\underline{A} = \underline{\nabla}\phi$. Find also the work done in moving an object in this field from $(0, 1, -1)$ to $\left(\frac{\pi}{4}, -1, 2\right)$.

   (c) State Green's theorem.
   Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the region bounded by the curve $y^2 = x$ and $y = x^2$.

4. Prove that the radial and transverse component of the acceleration of a particle in terms of the polar co-ordinates (r, θ) are

$$\ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \quad \text{respectively.}$$

   A light inextensible string of length $2a$ passes through a smooth ring at a point O, on a smooth horizontal table and two particles, each of mass m, attached to it's ends A and B. Initially the particles lie on the table with $OA = OB = a$ and AOB a straight line, the particle A is given a velocity u in a direction perpendicular to OA. Prove that if r and θ are the polar co-ordinates of A at a time t with respect to the origin, then

i. $2\dfrac{d^2r}{dt^2} - \dfrac{a^2u^2}{r^3} = 0,$

ii. $2r\dfrac{dr}{dt} = u\sqrt{2(r^2 - a^2)},$

iii. $r^2 = a^2 + \dfrac{1}{2}u^2t^2.$

Find the velocity of A at the instant when B reaches the origin at O.

5. A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\dfrac{dv}{dt}$ and $v\dfrac{d\psi}{dt}$, respectively.

A particle slides on a rough wire in the form of the cycloid $s = 4a\sin\psi$ which is fixed in a vertical plane with axis vertical and vertex downwards. It is projected from the vertex with speed u so that it comes to rest at the cusp ($\psi = \pi/2$). Show that

$$e^{\mu\pi} = \mu^2 + (\mu^2 + 1)u^2/4ag$$

where μ is the co-efficient of the friction.

Show also that when the particle slides from rest at the cusp, it will come to rest at the vertex if $e^{\mu\pi} = \mu^2$.

6. State the angular momentum principle for motion of a particle.

A right circular cone with a semi vertical angle α is fixed with its axis vertical and vertex downwards. A particle of mass m is held at the point A on the smooth inner surface of the cone at a distance $'a'$ from the axis of revolution. The particle is projected perpendicular to OA with velocity $'u'$, where O is the vertex of the cone. Show that the particle rises above the level of A if $u^2 > ag\cot\alpha$ and greatest reaction between the particle and the surface is

$$mg\left(\sin\alpha + \frac{u^2}{ag}\cos\alpha\right).$$