# EASTERN UNIVERSITY, SRI LANKA

## DEPARTMENT OF MATHEMATICS

## FIRST EXAMINATION IN SCIENCE - 2012/2013

## FIRST SEMESTER (Feb./Mar., 2015)

## AM 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

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Answer all questions

Time : Three hours

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1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove that

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}.$$

Hence show that

$$(\underline{a} \wedge \underline{b}) \cdot (\underline{c} \wedge \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c}).$$

(b) Find an equation for the plane passing through three points whose position vectors are given by $(2, -1, 1)$, $(3, 2, -1)$ and $(-1, 3, 2)$.

(c) Find the vector $\underline{x}$ and the scalar $\lambda$ which satisfy the equations

$$\underline{a} \wedge \underline{x} = \underline{b} + \lambda \underline{a}, \quad \underline{a} \cdot \underline{x} = 2,$$

where $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$.

2. Define the term *gradient* of a scalar field $\phi$.

   (a) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$.

      i. Show that $\nabla^2 r^n = n(n+1)r^{n-2}$, where $n$ is a constant;

      ii. Find the gradient of $\underline{\nabla}(\ln r)$ and $\nabla^2(\ln r)$.

   (b) Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ in the direction $2\underline{i} - 3\underline{j} + 6\underline{k}$.

   (c) Find the angle between the surface $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$.

3. (a) Define the following terms:

   - *a conservative vector field;*
   - *solenoidal vector field.*

   Show that $\underline{F} = (2x - y)\underline{i} + (2yz^2 - x)\underline{j} + (2y^2z - z)\underline{k}$ is conservative but not solenoidal.

   (b) Let $O = (0,0,0)$, $A = (1,0,0)$, $B = (1,2,0)$ and $C = (1,2,3)$. By considering the straight line path $OA, AB, BC$, find the line integral $\int_\gamma \underline{F} \cdot d\underline{r}$, where $\gamma$ is a path from $O$ to $C$.

   (c) State the Divergence theorem, and use it to evaluate $\int\int_S \underline{F} \cdot \underline{n}\, dS$, where $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and $S$ is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

4. Two particles each of mass $m$ are connected by a light in-extensible string and are lying on a smooth horizontal table with the string taut. The string passes through a small ring $O$ fixed to the table at a point distance '$a$' from the first particle. The first particle is given a horizontal velocity $v$ perpendicular to the string. Prove that its subsequent path until the second particle reaches $O$ is a polar equation given by $r = a\sec(\theta/\sqrt{2})$ relative to $O$. Prove also that, if the particle has reached a distance $r$ after time $t$, then $r^2 = a^2 + \frac{1}{2}v^2t^2$.

5. A particle moves in a plane with the velocity $v$ and the tangent to the path of the particle makes an angle $\psi$ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$, respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity $v_0$. The parachute exerts a drag opposing motion which is $k$ times the weight of the body, where $k$ is a constant. Neglecting the air resistance to the motion of the body, prove that if $v$ is the velocity of the body when its path is inclined an angle $\psi$ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$, the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

6. State the angular momentum principle.

A particle of mass $m$ moves on a smooth inner surface of a paraboloid of revolution $r^2 = 4az$ whose axis is vertical and vertex downwards, the path of the particle lies between the horizontal circle at $z = p$ and $z = q$. Show that the angular momentum of the particle above the axis is $m\sqrt{8agpq}$ and the velocity is $\sqrt{2g(p+q-z)}$, where $g$ is the acceleration due to gravity.

Show also that the reaction between the surface and the particle when the particle is at $z = p$ is given by

$$\frac{mg(a+q)}{\sqrt{a(p+q)}}.$$