EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 1994/95 & 95/96(Aug.'97)

I be a bounded function on [a, b]. Explain what is meant by the statement that

A. is connected, where I is an indexing set.

REPEAT

MT202 - METRIC SPACE AND RIEMANN INTEGRALS

Time: Two hours.

Answer four questions only.

1. Let (X, ρ) be a metric space and A, B be subsets of X. Define

- (a) the closure \overline{A} of A,
- (b) the interior A^0 of A,
- (c) the distance $\rho(x, A)$ of x from A.

Prove that

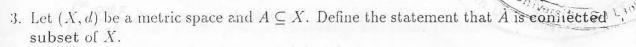
- (a) $(A \cap B)^0 = A^0 \cap B^0$,
- (b) $(A^0 \cup B^0) \subseteq (A \cup B)^0$,
- (c) $\overline{X} \setminus A = X \setminus A^0$, positive and Riemann integrable function on $A = A^0$
- (d) $X \setminus \overline{A} = (X \setminus A)^0$, a point of the mathematical
- (e) $x \in \overline{A}$ if and only if $\rho(x, A) = 0$.

Illustrate by means of an example that equality does not necessarily hold in part (b).

2. Let (X, ρ) be a metric space and A a subset of X.

Prove that

- (a) if X is complete and A is closed then A is complete;
- (b) if A is compact then A is closed and bounded;
- (c) every infinite subset of a compact set has a limit point;
- (d) if f is a continuous function on a metric space (X, ρ) , then image of a compact set is compact.



(a) Prove that a metric space (X, d) is connected if, and only if the only non-empty subset of X, which is both open and closed, is X itself.

Hence or otherwise prove that if A and B are connected subsets of ε metric space (X,d) such that $A \cap B \neq \Phi$, then $A \cup B$ is connected.

- (b) Prove that if $\{A_{\alpha}/\alpha \in I\}$ is a family of connected sets in X whose inte section is non-empty, then $\bigcup_{\alpha \in I} A_{\alpha}$ is connected, where I is an indexing set.
- 4. Let f be a bounded function on [a, b]. Explain what is meant by the statement that "f is Riemann integrable over [a, b]".
 - (a) With the usual notations, prove that a bounded function f on [a, b] is Riemann integrable if and only if for given $\epsilon > 0$, there exists a partition P of [a, b] such that

$$U(P, f) - L(P, f) < \epsilon.$$

- (b) Prove that if f is continuous on [a, b], then
 - i. f is Riemann integrable over [a, b];
 - ii. the function $F:[a,b] \longrightarrow \Re$ defined by $F(x) = \int_a^x f(t) dt$ is differentiable on [a,b] and $F'(x) = f(x) \quad \forall x \in [a,b]$;
 - iii. if $\phi:[a,b] \longrightarrow \Re$ is positive and Riemann integrable function over [a,b], then there exists a point c in [a,b] such that

$$\int_a^b f(x)\phi(x) \ dx = f(c) \int_a^b \phi(x) \ dx.$$

- 5. Prove or disprove each of the following statements. Justify your answers.
 - (a) In a metric space the union of a collection of open sets is open.
 - (b) If $f:[a,b]\to\Re$ is define by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{otherwise,} \end{cases}$$

then f is Riemann integrable over [a, b].

- (d) \overline{A} is the smallest closed set containing A.
- 6. Discuss the convergence of the following integrals.

(a)
$$\int_0^1 \frac{e^x}{\sqrt{x}} \, dx,$$

(b)
$$\int_0^\infty \frac{x}{5x^3 + 2x^2 + x + 5} dx$$
,

b) Plant Growth Movements

(c)
$$\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\tan x}} dx.$$