EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE (2003/2004)

June/July. 2005

SECOND SEMESTER

MT 102 - ANALYSIS I

Proper and Repeat

Answer **all** questions   Time: Three hours

1. (a) i. Define the terms "Supremum" and "Infimum" of a non-empty subset of $\mathbb{R}$.

   ii. State the completeness property of $\mathbb{R}$. [20]

   (b) Prove that a lower bound l of a non-empty bounded below subset S of $\mathbb{R}$ is the infimum of S if and only if for every $\epsilon > 0$, there exists $x \in S$ such that $x < l + \epsilon$.

   State the corresponding result for supremum. [30]

   (c) i. Let S_1 and S_2 be two non-empty bounded above subsets of $\mathbb{R}$ with $\text{Sup}S_1 = \alpha_1$ and $\text{Sup}S_2 = \alpha_2$. For any positive real numbers a and b, let the set $aS_1 + bS_2$ be defined by,

   $aS_1 + bS_2 = \{ax + by : x \in S_1, y \in S_2\}$.

   Prove the following:

   A. The set $aS_1 + bS_2$ is also bounded above

   B. $\text{Sup}(aS_1 + bS_2) = a\alpha_1 + b\alpha_2$. [35]

   ii. Find the Supremum and Infimum of the set $\left\{1 - \dfrac{1}{n} : n \in \mathbb{N}\right\}$, if they exist. [15]

2. (a) Define what is meant by each of the following terms applied to a sequence of real numbers.

i. bounded

ii. convergent

iii. monotone [30]

(b) Prove that, a monotone sequence (x_n) of real numbers is convergent if and only if it is bounded. [30]

(c) Let a sequence (x_n) be given by $x_{n+1}^2 - x_n - a = 0$ for all $n \geq 1$ and $x_1 > l$, where $a > 0$ and l is the positive root of the quadratic equation $x^2 - x - a = 0$. Prove that

i. $x_n > l$ for all $n \in \mathbb{N}$

ii. (x_n) is a strictly decreasing sequence.

Deduce that (x_n) converges and find its limit.

3. (a) i. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. What is meant by the function f has a limit $l \in \mathbb{R}$ at a point "a"($\in \mathbb{R}$).

ii. Show that if $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f(x)| = |l|$.

Is the converse of this result true? Justify your answer.

(b) i. Let $f : A(\subseteq \mathbb{R}) \to \mathbb{R}$, prove that $\lim_{x \to a^+} f(x) = l_1$ if and only if for every sequence (x_n) in A with $x_n \to a$ as $n \to \infty$ such that $x_n > a \ \forall n \in$ we have $f(x_n) \to l_1$ as $n \to \infty$.

State the corresponding result for $\lim_{x \to a^-} f(x) = l_2$.

Hence write the condition for the existance of $\lim_{x \to a} f(x) = l$.

ii. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \frac{1}{e^{\frac{1}{x}} + 1} \ \forall x \in \mathbb{R}$.

Prove that $\lim_{x \to 0} g(x)$ does not exist.

4. (a) i. Write the (ϵ, δ) definition of the statement that $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point "a"$(\in \mathbb{R})$.

ii. Show that, if f is continuous at 'a' and $f(a) < 0$ then there exists a $\delta > 0$ such that $3f(a) < 2f(x) < f(a)$ for all x satisfying $|x - a| < \delta$.

[40]

(b) i. If $f : [a, b](\subseteq \mathbb{R}) \to \mathbb{R}$ is continuous on $[a, b]$ then prove that it is bounded on $[a, b]$.

Is the converse part true? Justify your answer.

Discuss the result, if the domain of $f, [a, b]$ is changed to (a, b). [40]

ii. State the "Intermediate value" theorem.

Discuss the continuity of the function $f : \mathbb{R} \to \mathbb{R}$ on the interval $[0, 2]$ defined by

$$f(x) = \begin{cases} 4 & \text{if} \quad x \geq 1 \\ 2 & \text{if} \quad x < 1. \end{cases}$$

[20]

(Any results that used without proof should be clearly stated)

5. (a) State the Rolle's theorem and use it to prove the "Mean value" theorem.

If the function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $[a.b]$, differentiable on (a, b) and $f^{'}(x) = 0 \; \forall x \in [a, b]$, prove that f is a constant function on $[a, b]$.

[50]

(b) i. Let $a < b < c$ and suppose that $f : (a, c) \to \mathbb{R}$ is differentiable. Let $f^{'}(b) > 0$. Prove that if there exists $\delta > 0$ such that $0 < |x - b| < \delta$ then $\{f(x) - f(b)\}$ and $(x - b)$ have the same sign.

Hence show that $f(x) > f(b)$ if $b < x < b + \delta$ and $f(x) < f(b)$, if $b - \delta < x < b$. [30]

ii. Let $a < b < c$, $f : (a,c) \to \mathbb{R}$ be twice differentiable, $f'(b) = 0$ and $f''(b) > 0$. Show that if there if exists a $\delta > 0$ such that $b < x < b + \delta$, then $f'(x) > 0$ and hence that for the same δ, if $b < x < b + \delta$, then $f(x) > f(b)$. [20]

6. (a) Suppose that both real-valued functions f and g are continuous on $[a,b]$, differentiable on (a,b) and $g'(x) \neq 0 \ \forall \ x \in (a,b)$.

Prove that, for some $c \in (a,b)$,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

If $f(d) = g(d) = 0$ for some $d \in (a,b)$, deduce that $\lim_{x \to d} \frac{f'(x)}{g'(x)} = \lim_{x \to d} \frac{f(x)}{g(x)}$. [50]

(b) Evaluate the following limits

i. $\lim_{x \to 0} \left(\frac{\sqrt{1+x} - \frac{x}{2} - 1}{x^2} \right)$

ii. $\lim_{x \to 0} \left(\frac{\sqrt{1+x^2} - 1}{x \sin x} \right)$

iii. $\lim_{x \to \infty} x \log \left(1 + \frac{1}{x} \right)$. [45]