EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 2007/2008

FIRST SEMESTER (Dec/Jan, 2008/2009)

MT 201 - VECTOR SPACES AND MATRICES

Proper and Repeat

Answer all questions **Time: Three hours**

Q1. (a) Define what is meant by

(i) a vector space;

(ii) a subspace of a vector space.

(b) Let V be a vector space over a field $\mathbb{F}$. Prove that a non-empty subset W of V is a subspace of V if and only if $\alpha x + \beta y \in W$, for any $x, y \in V$ and $a, b \in \mathbb{F}$.

(c) Let $V = \left\{ \sum_{i=0}^{n} a_i X^i \; : \; a_i \in \mathbb{R}, n \in \mathbb{N} \right\}$ be the set of polynomials in one variable with real coefficients. The vector addition and scalar multiplication are defined as follows

$$\left(\sum_{i=0}^{m} a_i X^i \right) + \left(\sum_{j=0}^{n} b_j X^j \right) = \sum_{r} (a_r + b_r) X^r$$

where $a_r = 0$, if $r > m$ and $b_r = 0$, if $r > n$.

$$\alpha \left(\sum_{i} a_i X^i \right) = \sum_{i} \alpha a_i X^i, \; \forall \alpha \in \mathbb{R}.$$

Prove that V is a vector space over the field $\mathbb{R}$.

Q2. (a) Define the following

i. A linearly independent set of vectors;

ii. A basis for a vector space;

iii. Dimension of a vector space.

(b) Let V be an n-dimensional vector space.

Show that

i. A linearly independent set of vectors of V with n elements is a basis for V;

ii. Any linearly independent set of vectors of V may be extended to a basis for V;

iii. If L is a subspace of V, then there exists a subspace M of V such that $V = L \oplus M$;

(c) i. Let $\{u, v, w\}$ be a linearly independent subset of V. Prove or disprove, if $x = u + 2v + w,\ y = 2u + v + w$ and $z = u + v + w$ then the set $\{x, y, z\}$ is linearly dependent.

ii. Let $S = \{2, x, x - x^2, x + x^2\}$ be a subset of $\mathbb{P}_2$, where $\mathbb{P}_2 = \left\{ \sum_{i=0}^{2} a_i x^i \ :\ a_i \in \mathbb{R} \right\}$ be the set of all polynomials of degree ≤ 2 with real coefficients. Find the dimension of span of S.

Q3. (a) Define

(i) Range space $R(T)$;

(ii) Null space $N(T)$

of a linear transformation T from a vector space V into another vector space W.

Find $R(T)$, $N(T)$ of the linear transformation $T : \mathbb{R}^4 \to \mathbb{P}_2$, defined by

$$T(x_1, x_2, x_3, x_4) = x_1 + (x_2 + x_3)x + (x_2 - x_4)x^2.$$

Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation. Where $\mathbb{P}_2 = \left\{ \sum_{i=0}^{2} a_i x^i \ :\ a_i \in \mathbb{R} \right\}$ be the set of polynomials of degree ≤ 2.

(b) Let $T : \mathbb{R}^3 \to \mathbb{P}_3$, where $\mathbb{P}_3 = \left\{ \sum_{i=0}^{3} a_i x^i \ : \ a_i \in \mathbb{R} \right\}$ be the set of polynomials of degree ≤ 3, be a linear transformation defined by

$$T(\alpha, \beta, \gamma) = \alpha + (\beta + \gamma)x + (\gamma - \alpha)x^2 + \gamma x^3.$$

Find the matrix representation of T with respect to the basis $B_1 = \{(1,1,1), (1,2,3), (2,-1,1)\}$ and $B_2 = \{1 + x, x + x^2, x^2 + x^3, x^3\}$ of $\mathbb{R}^3$ and $\mathbb{P}_3$ respectively.

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (x + 2y, \ x + y + z, \ z)$ and let $B_1 = \{(1,0,0), \ (0,1,0), \ (0,0,1)\}$ and $B_2 = \{(1,1,0), \ (0,1,1), (1,0,1)\}$ be bases for $\mathbb{R}^3$. Find the matrix representation of T with respect to the basis B_2 by using the transition matrix.

Q4. (a) Define the following terms

(i) Rank of a matrix;

(ii) Echelon form of a matrix;

(iii) Row reduced echelon form of a matrix.

(b) Let A be an $m \times n$ matrix. Prove that

(i) row rank of A is equal to column rank of A;

(ii) if B is an $m \times n$ matrix obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

Q5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

(i) Cofactor A_{ij} of an element a_{ij};

(ii) Adjoint of A.

Prove that

$$A \cdot (adj A) = (adj A) \cdot A = det A \cdot I,$$

where I is the $n \times n$ identity matrix.

(b) Find the inverse of the matrix

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 3 & -2 \\ 2 & -1 & 3 \end{bmatrix}.$$

(c) Prove that if B is a matrix obtained from a matrix A by

(i) multiplying a row of A by a scalar $\alpha(\neq 0)$ then $|B| = \alpha\ |A|$.

(ii) interchanging two rows of A, then $|B| = -|A|$.

(d) Use mathematical induction or otherwise, prove the determinant of the $n \times n$ matrix

$$\begin{pmatrix} c & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & c & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & c & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & c & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & c \end{pmatrix},$$

where $c = 2\cos\theta$, which arises in the study of transverse vibration of strings by Langrange's method is $\dfrac{\sin[(n+1)\theta]}{\sin\theta}$.

Q6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Reduce the augmented matrix of the following system of linear equations to its row reduced echelon form and hence determine the conditions on non zero scalars $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 such that the system has

(i) a unique solution;

(ii) no solution;

(iii) more than one solution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

(b) Using row reduced echelon form, check wether the following system of equations is consistent, if so, find the solution(s).

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 &= -4 \\ 4x_1 - 2x_2 + x_3 + 2x_4 &= 20 \\ 3x_1 - x_2 + 3x_3 - 2x_4 &= 18 \\ 5x_1 - 3x_2 + 4x_3 - 3x_4 &= 32. \end{aligned}$$

(c) State and prove Crammer's rule for 3×3 matrix and use it to solve

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + x_2 - 10x_3 &= 4 \\ 2x_1 + 3x_2 - x_3 &= 2. \end{aligned}$$