EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 2007/2008

FIRST SEMESTER (Dec./Jan., 2008)

ST 201 - STATISTICAL INFERENCE - I

Answer all questions

Time : Two hours

Q1. A random sample $X_1, X_2, \cdots, X_n$ is taken from a Poisson distribution with mean λ and it is required to estimate $\theta = \lambda^2$.

(a) Show that the sample mean, $\bar{X}$, is a sufficient statistic for θ.

(b) Evaluate $E(\bar{X})$ and $E(\bar{X}^2)$ and hence find an unbiased estimator of θ based on $\bar{X}$.

(c) Find the Cramer - Rao lower bound for the variance of unbiased estimators of θ.

(d) Find the efficiency of your estimator in the case $n = 1$.

Q2. (a) Define

i. A maximum likelihood estimator.

ii. A method of moment estimator.

(b) A random sample $X_1, X_2, \cdots, X_n$ is obtained from a distribution with probability density function,

$$f(x) = \begin{cases} \dfrac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, & 0 \leq x < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

where $\alpha(> 0)$ and $\beta(> 0)$ are unknown parameters.

Estimate α and β by using the method of moments.

(c) Determine the maximum likelihood estimate for σ^2 in the following Rayleigh family distribution based on a random sample of size n:

$$f(x) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}, \quad x > 0$$

Q3. (a) Suppose that two independent random samples of n_1 and n_2 observations are selected from normal populations with means μ_i and variances σ_i^2, $i = 1, 2$. We wish to construct a confidence interval for the variance ratio $\frac{\sigma_1^2}{\sigma_2^2}$. Let s_i^2, $i = 1, 2$ be as defined below,

$$s_i^2 = \frac{\sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)^2}{n_i - 1}, \quad i = 1, 2.$$

Then find a confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$, with confidence coefficient $(1 - \alpha)$.

(b) A random sample of $n_1 = 10$ observations on breaking strength of a type of glass gave $s_1^2 = 2.31$ (measurements were made in pounds per square inch). An independent random sample of $n_2 = 16$ measurements on a second machine, but with the same kind of glass gave $s_2^2 = 3.68$. Estimate the true variance ratio, $\frac{\sigma_2^2}{\sigma_1^2}$ in 90% confidence interval.

(c) A factory operates with two machines of type A and one machine of type B. The weekly repair costs Y for the type A machines are normally distributed with mean μ_1 and variance σ^2. The weekly repair costs X for machines of type B are also normally distributed but with mean μ_2 and variance $3\sigma^2$. The expected repair cost per week for the factory is then $2\mu_1 + \mu_2$. If you are given a random sample $Y_1, Y_2, \cdots, Y_n$ on costs of type A machines and an independent random sample $X_1, X_2, \cdots, X_m$ on costs for type B machines, show how you would construct a 95% confidence interval for $2\mu_1 + \mu_2$. (Assume σ^2 is not known).

Q4. (a) Define the following terms

i. Unbiased estimate.

ii. Sufficiency.

iii. Consistency.

(b) Suppose $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 respectively. Suppose that the populations are normally distributed with $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Show that

$$\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{n}(Y_i - \bar{Y})^2}{(2n-2)}$$

is a consistent estimator of σ^2.

(c) Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with probability density function,

$$f(x, \theta) = \theta x^{\theta - 1}, \qquad 0 < x < 1, \ \ \theta > 0.$$

Show that $t_1 = \prod_{i=1}^{n} X_i$ is sufficient for θ.

(d) Suppose $Y_1, Y_2, \cdots, Y_n$ is a random sample from a population with probability density function

$$f(y) = \begin{cases} \dfrac{1}{\theta + 1} e^{-y/\theta+1}, & y > 0, \ \ \theta > -1 \\ 0, & \text{elsewhere.} \end{cases}$$

Suggest a suitable statistic to be used as an unbiased estimator for θ.