EASTERN UNIVERSITY, SRI LANKA
SPECIAL DEGREE EXAMINATION IN MATHEMATICS (2004/2005)
MARCH/APRIL, 2007
PART I
MT 406 – APPROXIMATION THEORY

Time: 3 Hours **Maximum Marks: 600**

Answer ALL Questions

I. (*a*) If K is a closed convex subset of R^n, then show that K possesses a unique point of minimum norm.

(*b*) Show, if X is a uniformly convex Banach space and $K \subset X$ is a closed convex set, that each $f \in X$ has a unique best approximation p* from K.

(*c*) Let X be a strictly convex normed space and $M \subset X$ be a finite dimensional subspace. Prove that each $f \in X$ has a unique best approximation from M. [25 + 40 + 35 = 100]

II. (*a*) Let $f \in C[a,b]$ and let $g_1, \ldots, g_n \in C[a,b]$ with $g_1, \ldots, g_n$ linearly independent. Define $\hat{x} = (g_1(x), \ldots, g_n(x))$, $x \in [a,b]$. Prove that for $P = \Sigma c_i g_i$ to be a best approximation, that is $c_1, c_2, \ldots, c_n$ to be such that the residual $r = f - \sum_{i=1}^{n} c_i g_i$ has minimum norm, it is necessary and sufficient that $\underline{0} \in Co\{r(x)\hat{x} : x \in [a,b] \text{ and } |r(x)| = \|r\|\}$.

(*b*) Let $\{g_1, g_2, \ldots, g_n\}$ form a Chebyshev system on [a,b]. Let $a \le x_0 < x_1 < x_2 < \ldots < x_n < b$ and $\lambda_0, \lambda_1, \ldots, \lambda_n \neq 0$. Prove that in order that $\underline{0} \in Co\{\lambda_0 \hat{x}_0, \lambda_1 \hat{x}_1, \ldots, \lambda_n \hat{x}_n\}$, it is necessary and sufficient that $\lambda_j \lambda_{j+1} < 0$, $j = 0,1,2,\ldots, n-1$. [55 + 45 = 100]

III. (*a*) Prove: $\min_{c_1, c_2, \ldots, c_{n-1}} \int_0^{\pi} \left| x - \sum_{k=1}^{n-1} c_k \sin(kx) \right| dx = \pi^2/(2n)$.

(*b*) Define the modulus of continuity of $f \in C_{2\pi}$ and, for $f \in C_{2\pi}$, prove that
$\mathcal{E}_n[f] \le (3/2)\, \omega(f; \frac{\pi}{n+1})$, n = 1,2,3,......

(*c*) Let $f \in C_{2\pi}$ and $0 < \alpha < 1$. Prove that f satisfies the condition that, for some B > 0, $|f(x) - f(y)| \le B|x - y|^{\alpha}$, for all $x,y \in [0,2\pi]$ if there exists A > 0 such that $\mathcal{E}_n[f] \le An^{-\alpha}$, $n \ge 1$.
[30 + 25 + 45 = 100]

IV. (*a*) Let $f \in C[-1,1]$ and let k be a positive integer and let $0 < \alpha < 1$. Assume that, for some $A > 0$, $\mathcal{E}_n[f] \le An^{-k-\alpha}$, $n \ge 1$. Show that $f^{(k)}$ exists and is continuous in (-1,1) and, given $0 < \delta < 1$, there exists B > 0 such that $|f^{(k)}(x) - f^{(k)}(y)| \le B|x - y|^{\alpha}$, for all $x,y \in [-1 + \delta, 1 - \delta]$.

(*b*) Let X be the space of continuous functions f: $[0,1] \to R$ with inner product

$(f,g) = \int_0^1 f(x)g(x)dx$. Let M be a finite dimensional subspace of X with basis

$\{x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_n}\}$. $\alpha_1, \alpha_2, \ldots, \alpha_n \ge 0$, distinct. Prove that the distance from x^m $(m \ge 0)$

IV. (*a*) Let f: $U \subset R^n \to R^m$ be a C^1 mapping where U is a neighborhood of the line segment L with end points a and b. Prove that $|f(b) - f(a)|_0 \le |b - a|_0 \max_{x \in L} \|f'(x)\|$.

(*b*) Suppose that the mapping f: $R^n \to R^n$ is C^1 in a neighborhood W of the point a, with the matrix f'(a) being nonsingular. Prove that f is locally invertible – i.e., there exist neighborhoods $U \subset W$ of a and V of b = f(a), and a one-to-one C^1 mapping g: $V \to W$ such that g(f(x)) = x for $x \in U$ and f(g(y)) = y for $y \in V$; and, in particular, prove that the local inverse g is the limit of the sequence $\{g_k\}_0^\infty$ of successive approximations, defined inductively by

$$g_0(y) = a,\ g_{k+1}(y) = g_k(y) - f'(a)^{-1}[f(g_k(y)) - y] \text{ for } y \in V.$$

(*c*) Let the C^1 mapping f: $R^2_{uv} \to R^2_{xy}$ be defined by the equations

$$x = u + (v + 2)^2 + 1$$
$$y = (u - 1)^2 + v + 1.$$

Let a = (1,–2). Is f invertible near a? If so, find a local inverse of f. [*25* + *35* + *40* = 100]

V. (*a*) State the General Implicit Mapping Theorem.

Solve $x^2 + \frac{1}{2} y^2 + z^3 - z^2 - 3/2 = 0$
$x^3 + y^3 - 3y + z + 3 = 0$ for y and z as functions of x in a neighborhood of (-1,1,0).

(*b*) Prove that every admissible function is integrable.

(*c*) Let f: $R^{m+n} = R^m \times R^n \to R$ be an integrable function such that, for each $x \in R^m$, the function f_x: $R^n \to R$, defined by $f_x(y) = f(x,y)$, is integrable. Given the contented sets $A \subset R^m$ and $B \subset R^n$, let F: $R^m \to R$ be defined by $F(x) = \int_B f_x = \int_B f(x,y)dy$. Then prove that F is integrable, and

$$\int_{A\times B} f = \int_A F = \int_A \left(\int_B f(x,y)dy \right) dx.$$

(*d*) Find the mass of the ellipsoidal ball $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ with the uniform density of unity.

[*30* + *20* + *30* + *20* = 100]

VI. (*a*) If f is a real-valued C^1 function on the open set $U \subset R^n$ and $\gamma:[a,b] \to U$ is a C^1 path, prove that $\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$.

(*b*) If α is a C^1 differential k-form on an open subset of R^n, prove that d(dα) = 0.

(*c*) If φ: $R^m \to R^n$ is a C^1 mapping and α is a C^1 differential k-form, show that d(φ*α) = φ*(dα).

(*d*) Let Q = [0,1] × [0,1] $\subset R^2$ and suppose φ: $Q \to R^3$ is defined by the equations

$$x = u + v,$$
$$y = u - v,$$
$$z = uv.$$

Then compute the surface integral $\int_\varphi x\,dy \wedge dz + y\,dx \wedge dz = \int_\varphi \alpha$ in two different methods you are aware of. [*20* + *20* + *30* +*30* = 100]