EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2002/2003

April./May.'2004

Repeat

MT 201 - VECTOR SPACES AND MATRICES

Answer four questions **Time: Two hours**

Q1. (a) Define what is meant by

(i) a vector space;

(ii) a subspace of a vector space.

Let V be a vector space over a field F and W be a non-empty subset of V. Prove that W is a subspace of V if and only if $ax + by \in W$ for every $x, y \in W$ and for every $a, b \in F$.

(b) Let W_1 and W_2 be two subspaces of a vector space V over a field F and let A_1 and A_2 be non-empty subsets of V. Show that

(i) $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2;

(ii) if A_1 spans W_1 and A_2 spans W_2 then $A_1 \cup A_2$ spans $W_1 + W_2$.

(c) Let V be the vector space of all functions from real field $\mathbb{R}$ into $\mathbb{R}$. Which of the following subsets are subspaces of V? Justify your answer.

(i) $W_1 = \{f \in V : f(3) = 0\}$

(ii) $W_2 = \{f \in V : f(7) = f(1)\}$

(iii) $W_3 = \{f \in V : f(-x) = f(x), \forall x \in \mathbb{R}\}$

(iv) $W_4 = \{f \in V : f(7) = 2 + f(1)\}$.

Q2. (a) Define the following:

i. A linearly independent set of vectors;

ii. A basis for a vector space;

iii. Dimension of a vector space.

(b) Let V be an n-dimensional vector space.

Show that:

i. A linearly independent set of vectors of V with n elements is a basis for V;

ii. Any linearly independent set of vectors of V may be extended as a basis for V;

iii. If L is a subspace of V, then there exists a subspace M of V such that $V = L \oplus M$;

iv. Extend the subset $\{(1, 2, -1, 1), (0, 1, 2, -1)\}$ to a basis for $\mathbb{R}^4$.

(State any results you may use)

Q3. (a) Define:

(i) Range space $R(T)$;

(ii) Null space $N(T)$

of a linear transformation T from a vector space V in to another vector space W.

Find $R(T)$, $N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by:

$$T(x, y, z) = (2x + y + 3z, 3x - y + z, -4x + 3y + z)$$

Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by:

$T(x, y, z) = (x+2y,\ x+y+z,\ z)$ and let $B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $B_2 = \{(1,1,0), (0,1,1), (1,0,1)\}$ be bases for $\mathbb{R}^3$.

Find:

(i) the matrix representation of T with respect to the basis B_1;

(ii) the matrix representation of T with respect to the basis B_2 by using the transition matrix;

(iii) the matrix representation of T with respect to the basis B_2 directly.

Q4. (a) Define the following terms

(i) Rank of a matrix;

(ii) Echelon form of a matrix;

(iii) Row reduced echelon form of a matrix.

(b) Let A be an $m \times n$ matrix. Prove that

(i) row rank of A is equal to column rank of A;

(ii) if B is an $m \times n$ matrix obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

Q5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

(i) Cofactor A_{ij} of an element a_{ij} ,

(ii) Adjoint of A.

Prove that

$$A \cdot (adj A) = (adj A) \cdot A = det A \cdot I$$

where I is the $n \times n$ identity matrix.

(b) If A and B are two $n \times n$ non-singular matrices, then prove that

(i) $adj(\alpha A) = \alpha^{n-1} \cdot adj A$ for every real number α,

(ii) $adj(AB) = (adj B)(adj A)$;

(iii) $adj(A^{-1}) = (adj A)^{-1}$;

(iv) $adj(adj A) = (det A)^{n-2} A$;

(v) $adj(adj(adj A)) = (det A)^{n^2-3n+3} A^{-1}$.

(c) Find the inverse of the matrix

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}.$$

Q6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Reduce the augmented matrix of the following system of linear equations to its row reduced echelon form and hence determine the conditions on a, b, c, d, e and f such that the system has;

(i) a unique solution;

(ii) no solution;

(iii) more than one solution.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f. \end{aligned}$$

(b) State and prove **Crammer's** rule for 3×3 matrix and use it to solve:

$$\begin{aligned} 2x_1 - 5x_2 + 2x_3 &= 7 \\ x_1 + 2x_2 - 4x_3 &= 3 \\ 3x_1 - 4x_2 - 6x_3 &= 5. \end{aligned}$$

(c) The system of equations,

$$\begin{aligned} 2x + 3y + z &= 5 \\ 3x + 2y - 4z + 7t &= k + 4 \\ x + y - z + 2t &= k - 1 \end{aligned}$$

is known to be consistent. Find the value of k and the general solution of the system.