

EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2002/2003 &

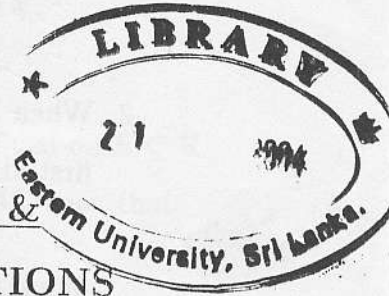
2002/2003(A) (Apr./May.'2004)

SECOND SEMESTER

(Repeat)

MT 204 - RIEMANN INTEGRAL &

SEQUENCE AND SERIES OF FUNCTIONS



Answer all questions

Time : Two hours

1. Let f be a real valued bounded function on $[a, b]$. Explain what is meant by the statement that " f is Riemann integrable over $[a, b]$."

(a) With the usual notations, prove that a bounded function f on $[a, b]$ is Riemann integrable if, and only if, for given $\epsilon > 0$ there exists a partition P such that

$$U(P, f) - L(P, f) < \epsilon.$$

(b) Let f be a Riemann integrable function on $[a, b]$. Prove that

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{(b-a)}{n} \sum_{k=1}^n f\left(a + k \frac{(b-a)}{n}\right).$$

Hence prove that

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} \right] = \ln 2.$$

(c) Let f be a real valued continuous function on a bounded interval $[a, b]$ and F be any function for which $F'(x) = f(x)$ for all $x \in [a, b]$. Prove that $\int_a^b f(x)dx = F(b) - F(a)$.

2. When is an integral $\int_a^b f(x)dx$ said to be an improper integral of the first kind, the second kind and the third kind?

What is meant by the statements "an improper integral of the first kind is convergent" and "an improper integral of the second kind is convergent"?

Discuss the convergence of the following:

i $\int_0^{\infty} \frac{dx}{\sqrt{x} \sqrt{1+4x^2}}$;

ii $\int_1^{\infty} \frac{\sin x}{x} dx$;

iii $\int_3^5 \frac{\ln x}{x-3} dx$.

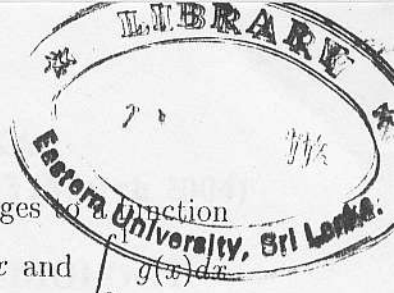
3. Define the term "Uniform convergence" of a sequence of functions.

(a) If $\{f_n(x)\}$ is a sequence of continuous functions which converges uniformly to f on $E \subseteq \mathbb{R}$ and if c is a limit point of E , show that

i. f is continuous on E ,

ii. $\lim_{x \rightarrow c} \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow c} f_n(x)$.

(b) i. Let $\{f_n\}$ be a sequence of functions that are integrable on $[a, b]$ and suppose that $\{f_n\}$ converges uniformly on $[a, b]$ to f . Prove that f is integrable and $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx$.



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- ii. Provide a sequence of functions $\{g_n\}$ converges to a function g on an interval $[0, 1]$ such that $\int_0^1 g_n(x) dx$ and $\int_0^1 g(x) dx$ exist and $\lim_{n \rightarrow \infty} \int_0^1 g_n(x) dx \neq \int_0^1 g(x) dx$.

4. (a) Let $\{f_n\}$ be a sequence of real valued functions defined on $E \subseteq \mathbb{R}$. Suppose that for each $n \in \mathbb{N}$, there is a constant M_n such that

$$|f_n(x)| \leq M_n \quad \text{for all } x \in E,$$

where $\sum M_n$ converges. Prove that $\sum f_n$ converges uniformly on E .

- (b) For each $n \in \mathbb{N}$, let $\{f_n\}$ be a sequence of real valued functions on (a, b) which has a derivative f'_n on (a, b) . Suppose that the series $\sum (f_n)$ converges for at least one point of (a, b) and that the series of derivatives $\sum (f'_n)$ converges uniformly on (a, b) . Prove that

i. there exists a real valued function f on (a, b) such that $\sum (f_n)$ converges uniformly on (a, b) to f .

ii. f has a derivative on (a, b) and $f' = \sum f'_n$.

- (c) Prove that, for $0 \leq x \leq 1$, the series $\sum_{n=1}^{\infty} \frac{e^{-nx}}{n^3}$ converges uniformly to a function f in $[0, 1]$.

Show that the series

$$\sum_{n=1}^{\infty} \frac{d}{dx} (e^{-nx}/n^3)$$

converges uniformly and $\sum_{n=1}^{\infty} f'_n(x) = f'(x)$.