EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2002/2003 & 2002/2003(A) (Apr./May.'2004)

SECOND SEMESTER

(Repeat)

MT 204 - RIEMANN INTEGRAL & SEQUENCE AND SERIES OF FUNCTIONS

Answer all questions

Time : Two hours

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1. Let f be a real valued bounded function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$.

   (a) With the usual notations, prove that a bounded function f on $[a, b]$ is Riemann integrable if, and only if, for given $\epsilon > 0$ there exists a partition P such that

   $$U(P, f) - L(P, f) < \epsilon.$$

   (b) Let f be a Riemann integrable function on $[a, b]$. Prove that

   $$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{(b-a)}{n} \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right).$$

   Hence prove that

   $$\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right] = \ln 2.$$

(c) Let f be a real valued continuous function on a bounded interval $[a, b]$ and F be any function for which $F'(x) = f(x)$ for all $x \in [a, b]$. Prove that $\int_a^b f(x)dx = F(b) - F(a)$.

2. When is an integral $\int_a^b f(x)dx$ said to be an improper integral of the first kind, the second kind and the third kind?

What is meant by the statements " an improper integral of the first kind is convergent" and "an improper integral of the second kind is convergent"?

Discuss the convergence of the following:

i $\int_0^{\infty} \frac{dx}{\sqrt{x}\,\sqrt{1+4x^2}}$;

ii $\int_1^{\infty} \frac{\sin x}{x}\, dx$;

iii $\int_3^5 \frac{\ln x}{x-3}\, dx$.

3. Define the term "Uniform convergence " of a sequence of functions.

(a) If $\{f_n(x)\}$ is a sequence of continuous functions which converges uniformly to f on $E \subseteq \mathbb{R}$ and if c is a limit point of E, show that

i. f is continuous on E,

ii. $\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x)$.

(b) i. Let $\{f_n\}$ be a sequence of functions that are integrable on $[a, b]$ and suppose that $\{f_n\}$ converges uniformly on $[a, b]$ to f. Prove that f is integrable and $\int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx$.

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ii. Provide a sequence of functions $\{g_n\}$ converges to a function g on an interval $[0,1]$ such that $\int_0^1 g_n(x)dx$ and $\int_0^1 g(x)dx$ exist and $\lim_{n\to\infty}\int_0^1 g_n(x)dx \neq \int_0^1 g(x)dx$.

4. (a) Let $\{f_n\}$ be a sequence of real valued functions defined on $E \subseteq \mathbb{R}$. Suppose that for each $n \in \mathbb{N}$, there is a constant M_n such that

$$|f_n(x)| \leq M_n \qquad \text{for all } x \in E,$$

where $\sum M_n$ converges. Prove that $\sum f_n$ converges uniformly on E.

(b) For each $n \in \mathbb{N}$, let $\{f_n\}$ be a sequence of real valued functions on (a,b) which has a derivative f_n' on (a,b). Suppose that the series $\sum(f_n)$ converges for at least one point of (a,b) and that the series of derivatives $\sum(f_n')$ converges uniformly on (a,b). Prove that

i. there exists a real valued function f on (a,b) such that $\sum(f_n)$ converges uniformly on (a,b) to f.

ii. f has a derivative on (a,b) and $f' = \sum f_n'$.

(c) Prove that, for $0 \leq x \leq 1$, the series $\sum_{n=1}^{\infty} \frac{e^{-nx}}{n^3}$ converges uniformly to a function f in $[0,1]$.

Show that the series

$$\sum_{n=1}^{\infty} \frac{d}{dx}\left(e^{-nx}/n^3\right)$$

converges uniformly and $\sum_{n=1}^{\infty} f_n'(x) = f'(x)$.