EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SPECIAL DEGREE EXAMINATION IN MATHEMATICS - 2008/2009
(December, 2010)
Part II
MT 401 - FUNCTIONAL ANALYSIS II

Answer all questions	**Time allowed: 3 hours**

1)

a. State the Hahn-Banach Theorem for normed linear spaces.
Let Y be a proper closed subspace of a normed linear space X.
Let $x_0 \in X \setminus Y$ be arbitrary and $\delta = \underset{y \in Y}{Inf} \|y - x_0\|$.

Prove that there exists a bounded linear functional f on X such that $\|f\| = 1, f(x_0) = \delta$ and $f(y) = 0$ for all $y \in Y$.

b. What do you mean by a normed linear space X is separable?
Prove that if the dual space X^* of a normed linear space X is separable, then X is separable.

2)

a. State the Baire's category theorem for Banach spaces and use it to prove the Uniform Boundedness Theorem.

b. With usual notations, define the canonical mapping $C : X \rightarrow X^{**}$, where X is a normed linear space and X^{**} is the double dual of X. Show that C is an isomorphism of X onto the normed space $R(C)$, the range of C, which preserve norm.

c. Let (x_n) be a sequence in a Banach space X such that $(f(x_n))$ is bounded for all $f \in X^*$. Show that $(\|x_n\|)$ is bounded.

3)

a. Let T be a bounded linear operator from a Banach space X **onto** a Banach space Y. Prove that T is an open mapping and further if T is bijective, then T^{-1} is a bounded linear operator. (You may assume without proof that the image $T(B_0)$ of the open unit ball $B_0 = B(0,1)$ in X contains an open ball about $0 \in Y$.)

b. Let X be the normed linear space whose points are sequences of complex numbers $x = (\xi_i)$ with only finitely many non-zero terms with the norm defined by $\|x\| = \underset{i}{Sup}|\xi_i|$. Let $T: X \to X$ be defined by

$T(x) = (\xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \ldots)$ Show that T is linear and bounded but T^{-1} is unbounded. Does this contradict the result you proved in part a? Justify your answer.

c. Let $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$ be Banach spaces. If there is a constant C such that $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$, show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

4)

a. Let $B[X,Y]$ be the set of all bounded linear operators from a normed linear space X into a normed linear space Y. Prove that if Y is a Banach space then $B[X,Y]$ is a Banach space with respect to the norm defined by $\|T\| = \underset{x \in X, x \neq 0}{Sup} \frac{\|Tx\|}{\|x\|}$.

b. With the usual notations, prove that there is a bijective linear operator between l^1 and l^∞ which preserves the norm.

5)

a. State and prove the Riesz representation theorem for a bounded linear functional on a Hilbert space.

b. Let X be an inner product space Prove that;

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{1}{2}(x+y)\right\|^2, \forall x,y,z\in X.$$

Give a geometric representation of the above identity when $X=\Re^2$.

c. Let X be an inner product space over C, and let $T:X\to X$ be a bounded linear operator such that $<Tx,x>=0$ for all $x\in X$. Prove that $T=0$.

6)

a. Let l^∞ be the linear space of all bounded sequences of complex numbers with the norm defined by $\|x\|=\underset{i}{Sup}|\xi_i|$, where $x=(\xi_i)$.

Define $T:l^\infty\to l^\infty$ by $T((\xi_1,\xi_2,\xi_3,\cdots))=(\xi_1,\frac{1}{2}\sum_{j=1}^{2}\xi_j,\frac{1}{3}\sum_{j=1}^{3}\xi_j,\cdots)$.

Prove that T is bounded linear operator and compute the norm $\|T\|$.

b. Let X and Y be normed spaces and let (T_n) be a sequence of bounded linear operators from X to Y. What do you mean by saying that

- (T_n) is strongly operator convergent; and
- (T_n) is weakly operator convergent.

Prove that strongly operator convergent always implies weakly operator convergent.but the converse is not generally true.

c. Let X be a Banach space and let (T_n) and (S_n) be two sequences of bounded linear operators on X such that $S_nT_m=T_mS_n$ for all $n,m\geq 1$. Assume that (T_n) and (S_n) are strongly operator convergent with limits T and S respectively. Prove that $TS=ST$.