EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SPECIAL DEGREE EXAMINATION IN MATHEMATICS -2008/2009
(December, 2010)
Part II
MT 402 - MEASURE THEORY

Answer all questions **Time allowed: 3 hours**

1) Let $f:[A,B]\to\Re$ be a bounded function, where $A,B\in\Re$ and $A<B$.
   a. With the usual notations, define the lower Riemann sum $s(f,\Delta)$ and upper Riemann sum $S(f,\Delta)$ of f corresponding to the dissection Δ of $[A,B]$.
   b. Suppose that Δ,Δ^{1} and Δ^{11} are dissections of the interval $[A,B]$ and that $\Delta^{1}\subseteq\Delta$. Prove the following:
      - $s(f,\Delta^{1})\le s(f,\Delta)$ and $S(f,\Delta)\le S(f,\Delta^{1})$
      - $s(f,\Delta^{1})\le S(f,\Delta^{11})$.
   c. What do you mean by f is Riemann integrable over $[A,B]$?
   Prove that the following conditions are equivalent.
      - f is Riemann integrable over $[A,B]$
      - Given $\in>0$, there exists a dissection Δ of $[A,B]$ such that $S(f,\Delta)-s(f,\Delta)<\in$.
   d. Suppose that f is Riemann integrable over $[A,B]$. Suppose further that $C\in[A,B]$ is such that $f(x)=g(x)$ for every $x\in[A,B]$ except possibly at $x=C$ Prove that g is Riemann integrable and $\int_A^B f(x)dx=\int_A^B g(x)dx$.
   e. Give an example of a function which is not Riemann integrable.

2) Explain what is meant by a step function on $\Re$.
   a. Let $f\in L(\Re)$. Prove that there exists a sequence (φ_n) of step functions such that $\varphi_n(x)\to f(x)$ almost everywhere in $\Re$, and that $\int_{\Re}|f-\varphi_n|\to 0$ as $n\to\infty$.
   b. If φ is a step function, show that $\int_{\Re}\varphi(x)\cos kx\,dx\to 0$ as $k\to\infty$.

c. Hence or otherwise, show that for $f \in L(\Re)$,

$$\int_{\Re} f(x)\cos kx\, dx \to 0 \quad \text{as} \quad k \to \infty .$$

3)

a. State the Fubini's theorem in $\Re^2$ and use it to prove the following.
Let $f \in M(\Re^2)$ and suppose that one of the integral

$$\int_0^\infty\left(\int_0^\infty |f(x,y)|dy\right)dx \,,\; \int_0^\infty\left(\int_0^\infty |f(x,y)|dx\right)dy \text{ exists. Prove that } f \in L^1(\Re^2).$$

b. Prove that, if $f(x,y) = ye^{-(1+x^2)y^2}$, for $(x,y) \in \Re^2$ then

$$\int_0^\infty\left(\int_0^\infty f(x,y)dy\right)dx = \int_0^\infty\left(\int_0^\infty f(x,y)dx\right)dy .$$

Deduce that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$.

4) Prove the following: (You may use any convergence theorem.)

a. $\int_0^\infty e^{-\alpha t}dt = 1/\alpha$, where $\alpha > 0$

b. $\int_0^\infty e^{-\alpha t}\cos\beta t dt = \dfrac{\alpha}{\alpha^2+\beta^2}$, where $\alpha, \beta > 0$

c. $\int_0^\infty \dfrac{\sin at}{e^t - 1}dt = \sum_{n=1}^{\infty}\dfrac{a}{a^2+n^2}$ where $a > 0$.

5)

a. State the Monotone Convergence Theorem and Dominated Convergence theorems in $L(I)$.

b. Suppose that $I = [A, \infty)$, where $A \in \Re$. Suppose further that the function $f: I \to \Re$ satisfies the following conditions:

- $f \in L([A,B])$ for every real number $B \geq A$.
- There exists a constant $M > 0$ such that $\int_A^B |f(x)|dx \leq M$ for every real number $B \geq A$.

Prove that

$f \in L(I)$, the limit $\lim\limits_{B\to\infty}\int_A^B f(x)dx$ exists and $\int_A^\infty f(x)dx = \lim\limits_{B\to\infty}\int_A^B f(x)dx$.

c. Let $f:[0,\infty)\to\Re$ be defined by $f(x)=n^{-1}\sin \pi x$, for every $x\in[n-1,n)$. Prove the following:

- $\lim\limits_{B\to\infty}\int_0^B f(x)dx=\dfrac{2\log 2}{\pi}$;
- $\int_0^B |f(x)dx|$ is not bounded as $B\to\infty$;
- $f\notin L([0,\infty))$.

6) Prove the following:

a. Every open set $G\subseteq\Re$ is a countable union of pair wise disjoint open intervals in $\Re$.

b. Suppose that $I\subseteq\Re$ is an interval and that $f:I\to\Re$ is given. If there exists a sequence (f_n) in $M(I)$, the set of all measurable functions on I, such that $f_n(x)\to f(x)$ as $n\to\infty$ for almost all $x\in I$, then $f\in M(I)$.

c. A subset S of $\Re$ has measure zero if and only if the following two conditions are satisfied:

- $\Psi_S\in L(\Re)$; and
- $\int_{\Re}\Psi_S(x)dx=0$.

d. There exist sequences $(g_n),(h_n)$ in $L(\Re)$ such that $g_n\to 0$ almost everywhere but $\int_{\Re}|g_n|$ does not converges to 0 as $n\to\infty$ **and** $\int_{\Re}|h_n|\to 0$ but (h_n) does not converge almost everywhere to 0 as $n\to\infty$.