EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SPECIAL DEGREE EXAMINATION IN MATHEMATICS
ACADEMIC YEAR - 2008/2009 (December, 2010)
Part II
MT 407 - RING THEORY

Answer all questions. **Time allowed: Three hours**

1. (a) Prove that a field has no non-zero proper ideals. [20 Marks]

   (b) Prove that if a commutative ring with unity has no non-zero proper ideal, then it is a field. [20 Marks]

   (c) Let R be a commutative ring with unity. Prove the following:

      i. If M is a maximal ideal of R, then R/M is a field. [35 marks]

      ii. If P is a prime ideal of R if and only if R/P is an integral domain. [20 Marks]

      iii. Every maximal ideal is a prime ideal. [05 Marks]

2. (a) Show that every finite integral domain is a field. [30 Marks]

   (b) An element a of a ring R is called *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$. Show that the zero element is the only nilpotent element in an integral domain. [15 Marks]

   (c) If R is a ring with no zero divisors, then show that the equation $ax = b$ with $a \neq 0$ has at most one solution in R. [20 Marks]

   (d) Let $\Phi : R \longrightarrow R'$ be a homomorphism between rings R and R'. Prove the following:

      i. If I is an ideal of R, then $\Phi(I)$ is an ideal of $\Phi(R)$. [20 Marks]

      ii. If R is a ring with unity and $\Phi(1) \neq 0'$, then $\Phi(1)$ is the unity of the ring $\Phi(R)$. [15 Marks]

3. (a) Prove that every strictly ascending chain of ideals in a Principal Ideal Domain (PID) is of finite length, that is the ascending chain condition (ACC) holds for ideals in a PID. [30 Marks]

   (b) Let D be a PID. Show that every element that is neither 0 nor a unit in D is a product of irreducible elements of D. [30 Marks]

(c) Prove that every PID is a Unique Factorization Domain (UFD).

[30 marks]

(d) Show that the integral domain $\mathbb{Z}$ is a UFD. [10 Marks]

4. Let F be a field.

(a) State and prove the Division Algorithm for $F[x]$. [40 Marks]

(b) Show that a non-zero polynomial $f(x) \in F[x]$ of degree n can have at most n zeros in F. [30 Marks]

(c) Show that every ideal in $F[x]$ is principal. Hence deduce that $F[x]$ is a principal ideal domain (PID). [30 Marks]

5. (a) Let $\Phi : M \longrightarrow N$ be a module homomorphism. Prove that Φ is a **monomorphism** if and only if the kernal of Φ is the zero ideal.

[30 marks]

(b) Let R be a ring with identity and let M be a left R-module. Let I be an ideal of R such that $am = 0$ for all $a \in I$ and all $m \in M$. Show that M is a left (R/I)-module by an action defined by $(r + I).m = rm$. [30 Marks]

(c) Let R be a ring with identity and let M be a left R-module. Prove the following:

i. If M is a **Notherian left R-module,** then every non-empty set of **left** R- submodules of M contains a maximal element under inclusion. [20 Marks]

ii. If every left R-submodule of M is finitely generated, then show that M is a Notherian left R-module. [20 Marks]

6. Let R be a ring with identity.

(a) Prove that the R-module M is the direct sum of the family of submodules N_i, $i \in I$ if and only if for any family of homomorphism $\Phi_i : N_i \longrightarrow M'$ of the R-modules N_i, $i \in I$ into an R-module M', there exists a unique homomorphism $\Phi : M \longrightarrow M'$ which extends each Φ_i, that is for which $\Phi_i = \Phi \circ \eta_i$, $i \in I$ with η_i the inclusion of N_i in M. [40 Marks]

(b) If the subset X of the R-module M is such that every mapping of X into any R-module M' extends uniquely to a homomorphism of M into M', then show that X is a basis on M. [40 Marks]

(c) If X is a basis of the free R-module M, then show that every mapping X into any R-module M' can be extended in one and only one way to a homomorphism of M into M'. [20 Marks]