EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SPECIAL DEGREE EXAMINATION IN MATHEMATICS
ACADEMIC YEAR - 2008/2009 (December, 2010)
Part II
MT 410 - NUMERICAL LINEAR ALGEBRA

Answer all questions. **Time allowed: Three hours**

1. (a) Prove that an $n \times n$ matrix A has a unique LU factorization when the leading principal submatrices of order $r (\leq n)$, are nonsingular for $r = 1, 2, \ldots n - 1$. Hence, show that there exists a unit lower triangular matrix M and a diagonal matrix D such that $A = LDM^T$.

   (b) Deduce from (a) that if A is symmetric then $A = LDL^T$. Determine L and D such that

$$LDL^T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & -2 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

2. (a) Show that an elementary Hermitian matrix $H(\omega)$ defined by

$$H(\omega) = I - 2\omega\omega^H, \quad \omega^H\omega = 1 \quad \text{or} \quad \omega = 0,$$

   where ω is an n-column vector and $\omega^H = \bar{\omega}^T$ is Hermitian and unitary.

   (b) Let x and y be given n-column vectors such that $x^H x = y^H y$ and $x^H y = y^H x$. Show that there exists an elementary Hermitian matrix $H(\omega)$ such that $y = H(\omega)x$. Hence, show that for any $x \in \mathbb{C}^n$, there is an $n \times n$ elementary Hermitian matrix $H(\omega)$ such that $H(\omega)x = ke_1$, where $|k| = ||x||_2$ and

$e_1 = (1, 0, 0, \ldots, 0)^T \in \mathbb{R}^n$.

(c) Use the previous part to find an upper triangular matrix U such that $HA = U$, where H is a product of elementary Hermitian matrices and

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$

Hence, solve $Ax = (1, 0, 0)^T$.

3. (a) Consider an iteration of the form

$$Mx^{(r+1)} = Nx^{(r)} + b$$

for solving a linear system $Ax = b$, where A is an $n \times n$ nonsingular matrix and $A = M - N$. Define the choices of M and N that give the Jacobi iteration and the Gauss-Seidel iteration.

(b) Prove that if M is nonsingular and the spectral radius $\rho(M^{-1}N) < 1$ then the iterates $x^{(r)}$ given in (a) converges to $x = A^{-1}b$ for any $x^{(0)}$. (You may assume without proof that $\lim_{r\to\infty} B^r = 0$ if $\rho(B) < 1$.)

(c) Prove that $\rho(B) \leq ||B||$ for an $n \times n$ matrix and any subordinate matrix norm.

Hence, show that the Jacobi iteration converges if A is strictly diagonally dominant by rows, that is,

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1,\ 2,\ \ldots,\ n.$$

4. (a) Define the term *Upper Hessenberg* as applied to an $n \times n$ matrix A. Show that there exists a nonsingular matrix S, a product of elementary permutation matrices and elementary lower triangular matrices, such that $S^{-1}AS$ is an upper Hessenberg matrix.

(b) Find and upper Hessenberg matrix U and a nonsingular matrix S such that

$$SUS^{-1} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

5. (a) Let A be an $n \times n$ real symmetric matric with eigenvalues λ_i satisfying $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$ and corresponding orthomormal eigenvectors $z_1, z_2, \ldots, z_n$. Consider the Power method

$$x^{(r+1)} = \frac{1}{k_r} A x^{(r)}, \quad r = 0, 1, 2, \ldots \qquad (1)$$

where k_r is the component of $Ax^{(r)}$ of maximum modulus, applied to A with starting vector

$$x^{(0)} = \sum_{i=1}^{n} \alpha_i z_i, \quad \alpha_1 \neq 0.$$

Show that for some nonzero scalar β_r,

$$x^{(r)} = \beta_r \left(z_1 + \sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} \left(\frac{\lambda_i}{\lambda_1} \right)^r z_i \right).$$

Evaluate $||\beta_r^{-1} x^{(r)} - z_1||_2^2$. Hence explain the behaviour of $x^{(r)}$ and μ_r as $r \to \infty$,

where $\mu_r = \dfrac{x^{(r)T} A x^{(r)}}{x^{(r)T} x^{(r)}}, \quad r = 0, 1, 2, \ldots$

(b) Starting with $x^{(0)} = (0,\ 0,\ 1)^T$, obtain μ_2 by applying (1) to the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & -3 \\ 0 & -3 & 8 \end{pmatrix}.$$

6. (a) Suppose that the dominant eigenvalue λ_1 and corresponding eigenvectors z_1 of an $n \times n$ matrix A have been computed by the Power method.

i. Show that there is a nonsingular matrix S, a product of elementary permutation matrix and elementary lower triangular matrix, such that

$$A = S^{-1} \left(\begin{array}{c|c} \lambda_1 & \gamma^T \\ \hline 0 & B \end{array} \right) S$$

where B is an $(n-1) \times (n-1)$ matrix and γ is an $(n-1)$-column vector.

ii. Let z_2 be the eigenvector of A corresponding to the next dominant eigenvalue λ_2 and let y be the eigenvector of B corresponding to λ_2. Show that

$$(\lambda_1 - \lambda_2)\alpha + \gamma^T y = 0 \text{ and } z_2 = S^{-1} \begin{pmatrix} \alpha \\ y \end{pmatrix},$$

where α a scalar.

(b) It is given that the matrix

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{pmatrix}$$

has dominant eigenvalue 11 with corresponding eigenvector (0.5, 1.0, 0.75). Obtain the remaining eigenvalues of A.