EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2002/2003)

(Feb./Mar.'2004)

MT 301 - GROUP THEORY

REPEAT

Answer Five questions only **Time: Three hours**

1. State and prove Lagrange's theorem for a finite group G. [25]

   (a) In a group G, H and K are different subgroups of order p, p is prime. Show that $H \cap K = \{e\}$, where e is the identity element of G. [15]

   (b) Prove that in a finite group G, the order of each element divides order of G. Hence prove that $x^{|G|} = e, \ \forall\, x \in G$. [15]

   (c) Let G be a non-abelian group of order 20. Prove that G contains atleast one element of order 5 or 10. [15]

   (d) i. Let G be a group of order 27. Prove that G contains a sub group of order 3. [15]

   ii. Suppose that H, K are unequal subgroups of G, each of order 16. Prove that $24 \leq | H \cup K | \leq 31$. [15]

2. (a) What is meant by saying that a subgroup of a group is normal? [05]

i. Let H and K be two normal subgroups of a group G. Prove that $H \cap K$ is a normal subgroup of G [10]

ii. Prove that every subgroup of an abelian group is a normal subgroup. [10]

(b) With usual notations prove that

i. $N(H) \leq G$; [15]

ii. $H \trianglelefteq N(H)$; [15]

iii. $N(H)$ is the largest subgroup of G in which H is normal. [10]

(c) i. Let H be a subgroup of a group G such that $x^2 \in H$ for every x in G. Prove that $H \trianglelefteq G$ and G/H is abelian. [20]

ii. Show that a group in which all the m^{th} powers commute with each other and all the n^{th} powers commute with each other, m and n relatively prime, is abelian. [15]

(**Hint:**If m,n are relatively prime there exist integers x and y such that $xm + yn = 1$.)

3. (a) State and prove the first isomorphism theorem. [25]

(b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove that

i. $K \trianglelefteq H$; [05]

ii. $H/K \trianglelefteq G/K$; [20]

iii. $\dfrac{G/K}{H/K} \cong G/H$. [20]

(c) From second isomorphism theorem deduce that $|HK| = \frac{|H||K|}{|H \cap K|}$ where $H \leq G,\ K \trianglelefteq G$. [15]

Hence deduce that, if G is a finite group with a normal subgroup N such that $(|N|, |G/N|) = 1$, then N is the unique subgroup of G of order $|N|$. [15]

4. (a) Define the following terms as applied to a group G.

   i. commutator of two elements a, b of G; [10]

   ii. commutator subgroup (G'); [10]

   iii. internal direct product of two subgroups of G. [10]

(b) Prove that

   i. $G' \trianglelefteq G$; [15]

   ii. G/G' is abelian. [10]

(c) i. Let H and K be two subgroups of a group G, then prove that $G = H \otimes K$ if and only if

   A. each $x \in G$ can be uniquely expressed in the form $x = hk$, where $h \in H, k \in K$.

   B. $hk = kh$ for any $h \in H, k \in K$. [25]

   ii. Give an example to show that a group cannot always be expressed as the internal direct product of two non-trivial normal subgroups. [20]

(c) From second isomorphism theorem deduce that $| HK | = \frac{|H||K|}{|H \cap K|}$ where $H \leq G,\ K \trianglelefteq G$. [15]

Hence deduce that, if G is a finite group with a normal subgroup N such that $(| N |, | G/N |) = 1$, then N is the unique subgroup of G of order $| N |$. [15]

4. (a) Define the following terms as applied to a group G.

i. commutator of two elements a, b of G; [10]

ii. commutator subgroup (G'); [10]

iii. internal direct product of two subgroups of G. [10]

(b) Prove that

i. $G' \trianglelefteq G$; [15]

ii. G/G' is abelian. [10]

(c) i. Let H and K be two subgroups of a group G, then prove that $G = H \otimes K$ if and only if

A. each $x \in G$ can be uniquely expressed in the form $x = hk$, where $h \in H, k \in K$.

B. $hk = kh$ for any $h \in H, k \in K$. [25]

ii. Give an example to show that a group cannot always be expressed as the internal direct product of two non-trivial normal subgroups. [20]

5. Define the terms " automorphism" and "inner automorphism" of a group G. [10]

Let **Aut**G be the set of all automorphisms of G and let **Inn**G be the set of all inner automorphisms of G.

(a) Show that

i. **Aut**G is a group under composition of maps; [20]

ii. **Inn**G is a normal subgroup of **Aut**G. [20]

(b) If H is a subgroup of G, prove that $N(H)/Z(H) \cong \mathbf{InnG}$, [20]

Hence deduce that $G/Z(G) \cong \mathbf{InnG}$. [10]

Where, $N(H) = \{x \in H \mid xH = Hx\}$ and

$Z(H) = \{a \in H \mid ax = xa \; \forall x \in H\}$.

(c) If $G = \{a, b\}$, find **Aut**G for each of the binary operations " $*$ "and " $\times$ " defined by,

i. $a * a = a, \; a * b = b, \; b * a = b, \; b * b = a$;

ii. $a \times a = a, \; a \times b = b, \; b \times a = a, \; b \times b = b$. [20]

6. Define the following terms as applied to a group.

* Permutation;
* Cycle of order r;
* Transposition. [15]

(a) Prove that the permutation group on n symbols (s_n) is a finite group of order $n!$. [15]

Is it true that s_n is abelian for $n > 2$? Justify your answer. [10]

(b) Prove that every permutation in s_n can be expressed as a product of transpositions. [20]

(c) Prove that the set of even permutations forms a normal subgroup of s_n. [20]

(d) Prove with the usual notations that $A_n = s_n$ implies $n = 1$. [20]

7. What is meant by a conjugate class in a group? [10]

Write down the class equation of a finite group G. [05]

Hence or otherwise prove that

(a) i. If the order of G is p^n, where p is a prime number, then centre of G is non-trivial. [25]

ii. If the order of G is p^2, where p is prime number then G is abelian. [20]

(b) If G be a group of order 27, deduce that

i. G has a non-trivial centre $Z(G)$; [10]

ii. If G is non-abelian then order of the centre of G is 3. [10]

(c) Let G be a group containing an element of finite order $n > 1$ and exactly two conjugate classes. Prove that $| G |= 2$. [20]

8. Define the term $p-$group. [10]

(a) Prove that homomorphic image of a $p-$group is a $p-$group. [20]

(b) Let G be a finite abelian group and p be a prime number such that p is a divisor of the order of G. Prove that G has an element of order p. [40]

(c) "If G is a finite group, p a prime, and p^r the highest power of p dividing the order of G, then there is a subgroup of G of order p^r ".

Using the above fact or otherwise, prove that a finite group G is a $p-$group if and only if every element of G has order a power of p. [30]