EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE

(2002/2003 & 2002/2003(A))

SECOND SEMESTER(Feb./Mar.'2004)

MT 301 - GROUP THEORY

Answer all questions **Time: Three hours**

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1. State and prove Lagrange's theorem for a finite group G. [25]

   (a) In a group G, H and K are different subgroups of order p, p is prime. Show that $H \cap K = \{e\}$, where e is the identity element of G. [15]

   (b) Prove that in a finite group G, the order of each element divides order of G. Hence prove that $x^{|G|} = e$, $\forall\, x \in G$. [15]

   (c) Let G be a non-abelian group of order 20. Prove that G contains atleast one element of order 5 or 10. [25]

   (d) Let G be a group of order 27. Prove that G contains a sub group of order 3. [20]

2. (a) State and prove the first isomorphism theorem. [40]

(b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove that

i. $K \trianglelefteq H$; [10]

ii. $H/K \trianglelefteq G/K$; [20]

iii. $\dfrac{G/K}{H/K} \cong G/H$. [30]

3. (a) Define the following terms as applied to a group G.

i. commutator of two elements a, b of G ; [10]

ii. commutator subgroup (G') of G; [10]

iii. internal direct product of two subgroups of G. [10]

(b) Prove that

i. $G' \trianglelefteq G$; [15]

ii. G/G' is abelian. [10]

(c) i. Let H and K be two subgroups of a group G, prove that $G = H \otimes K$ if and only if

A. each $x \in G$ can be uniquely expressed in the form $x = hk$, where $h \in H, k \in K$.

B. $hk = kh$ for any $h \in H, k \in K$. [25]

ii. Give an example to show that a group cannot always be expressed as the internal direct product of two non-trivial normal subgroups. [20]

4. Define the terms " automorphism" and "inner automorphism" of a group G. [10]

Let $\mathbf{Aut}G$ be the set of all automorphisms of G and let $\mathbf{Inn}G$ be the set of all inner automorphisms of G.

(a) Show that

i. $\mathbf{Aut}G$ is a group under composition of maps; [20]

ii. $\mathbf{Inn}G$ is a normal subgroup of $\mathbf{Aut}G$. [20]

(b) If H is a subgroup of G, prove that $N(H)/Z(H) \cong \mathbf{InnG}$, [20]

Hence deduce that $G/Z(G) \cong \mathbf{InnG}$. [10]

Where, $N(H) = \{x \in H \mid xH = Hx\}$ and

$Z(H) = \{a \in H \mid ax = xa \ \forall x \in H\}$.

(c) If $G = \{a, b\}$, find $\mathbf{Aut}G$ for each of the binary operations " $*$ "and " $\times$ " defined by,

i. $a * a = a,\ a * b = b,\ b * a = b,\ b * b = a$;

ii. $a \times a = a,\ a \times b = b,\ b \times a = a,\ b \times b = b$. [20]

5. Define the following terms as applied to a group.

* Permutation;
* Cycle of order r;
* Transposition. [15]

(a) Prove that the permutation group on n symbols (s_n) is a finite group of order $n!$. [15]

Is it true that s_n is abelian for $n > 2$? Justify your answer. [15]

(b) Prove that every permutation in s_n can be expressed as a product of transpositions. [35]

(c) Prove with the usual notations that $A_n = s_n$ implies $n = 1$. [20]

6. Define the term $p-$group. [10]

(a) Prove that homomorphic image of a $p-$group is a $p-$group. [20]

(b) Let G be a finite abelian group and p be a prime number such that p is a divisor of the order of G. Prove that G has an element of order p. [40]

(c) "If G is a finite group, p a prime, and p^r the highest power of p dividing the order of G, then there is a subgroup of G of order p^r ".

Using the above fact or otherwise, prove that a finite group G is a $p-$group if and only if every element of G has order a power of p. [30]