# EASTERN UNIVERSITY, SRI LANKA

## SPECIAL DEGREE EXAMINATION IN MATHEMATICS

## (2001/2002) (Jan./Feb.2004)

## MT 410-NUMERICAL LINEAR ALGEBRA

You should answer all questions. Time allowed is **THREE** hours only. Each question carries **ONE HUNDRED** marks. The numbers beside the questions indicate the approximate marks that can be gained from the corresponding parts of the questions.

1. (a) Define the terms "positive definite" and "elementary lower-triangular" as applied to an $n \times n$ Hermitian matrix $A$.

[10]

(b) Prove that a positive definite matrix can be expressed as $A = LU$, where $L$ is a unit lower triangular matrix and $U$ is an upper triangular matrix.

[25]

(c) Show that a Hermitian matrix $A$ is positive definite if, and only if $A = GG^H$, where $G$ is a non-singular lower triangular matrix. [30]

Determine $G$ such that

$$GG^H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 4 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}.$$

[35]

2. (a) Define the terms "unitary matrix" and "elementary Hermitian matrix ". [10]

(b) Show that, for any real vector $x$, there is a real elementary Hermitian matrix $H(w)$ such that $H(w)x = ce_1$, where $c = x^T x$ and $e_1 = (1, 0, 0, \ldots, 0)^T$. What is the optimal choice of the sign of $c$ for the computation of $w$? [30]

(c) Determine an upper triangular matrix $U$ such that $HA = U$, where $H$ is a product of elementary Hermitian matrices and

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 4 & -1 \\ 2 & 5 & 0 \end{bmatrix}$$

making the optimal choice of sign in each stage of process. Hence solve the system $Ax = b$, where $b = (5, 0, -1)^T$ [60]

3. (a) Define the phrase strictly diagonally dominant applied to an $n \times n$ matrix $A$.

[10]

(b) Let $A = I - L - U$ be strictly diagonally dominant, where $L$ is strictly lower triangular and $U$ is strictly upper triangular. For arbitrary $x^{(0)}$, a sequence $\{x^{(r)}\}$ is defined by

$$x^{(r+1)} = (I - wL)^{-1}\{wb + [(1-w)I + wU]x^{(r)}\}, \quad r = 0, 1, 2, 3 \ldots .$$

Show that $x - x^{(r+1)} = M(x - x^{(r)})$, $r = 0, 1, 2, \ldots$, where $M = (I - wL)^{-1}[(1-w)I + wU]$ and $Ax = b$. State a necessary and sufficient condition for $\{x^{(r)}\}$ to converge to $x$. [15]

(c) Let $0 < w \leq 1$ and let $\lambda$ be any complex number with $|\lambda| \geq 1$. Show that $|\lambda + w - 1| \geq |w\lambda| \geq w$. Deduce that if $\lambda$ is any eigenvalue of $M$, then $|\lambda| < 1$. [35]

(d) The following equations are to be solved by successive over-relaxation with a relaxation parameter 1.1.

Starting with $x^{(0)} = 0$, obtain $x^{(1)}$, $x^{(2)}$ and bound for $\| x - x^{(2)} \|_\infty$.

$$\begin{bmatrix} 11 & 1 & 0 & 0 \\ 1 & 11 & 1 & 0 \\ 0 & 1 & 11 & 2 \\ 0 & 0 & 2 & 11 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

[40]

4. (a) Define the term "upper Hessenberg matrix." [10]

   (b) i. Let $A$ be an $n \times n$ matrix. Describe how a non-singular matrix $S$, a product of elementary lower triangular matrices and elementary permutation matrices, can be obtained so that $S^{-1}AS$ is an upper Hessenberg matrix. [35]

   (c) Given

$$A = \begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & -1 & 2 & 1 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$

   find an upper Hessenberg matrix $S^{-1}AS$, where $S$ is a product of elementary permutation matrices and elementary lower triangular matrices. [55]

5. (a) Let $A$ be an $n \times n$ symmetric positive definite matrix. Show that the solution of the system $Ax = b$ is equivalent to the unique minimum of the function

$$F(y) := \frac{1}{2} y^T A y - y^T b.$$

[20]

   (b) For given initial iterate $x_0$, the $k$th iterate $x_k$ is given by

$$x_k = x_{k-1} + \alpha p_k.$$

   where $p_k$ is the search direction to be chosen such that $p_k^T r_{k-1} \neq 0$, $r_k = b - Ax_k$ is residual. Show that

$$\alpha = \frac{p_k^T r_{k-1}}{p_k^T A p_k}$$

minimizes the function $F(x_{k-1} + \alpha p_k)$ with respect to $\alpha$. [20]

(c) Let $A$ be an $n \times n$ symmetric positive definite matrix and $b \in \mathbb{R}^n$. The Conjugate Gradient (CG)iterative method for solving the system $Ax = b$ is given by, for given initial iterate $x_0 = 0$,

$$
\begin{aligned}
\text{Set} \quad p_0 &= r_0. \\
\text{While} \quad r_k &\neq 0, \\
\alpha_k &= \frac{r_k^T r_{k-1}}{p_k^T A p_k}, & \text{(CG1)} \\
x_{k+1} &= x_k + \alpha_k p_k, & \text{(CG2)} \\
r_{k+1} &= r_k - \alpha_k A p_k, & \text{(CG3)} \\
\beta_{k+1} &= \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}, & \text{(CG4)} \\
p_{k+1} &= r_{k+1} + \beta_{k+1} p_k. & \text{(CG5)}
\end{aligned}
$$

Show that

$$< r_0, r_1, \cdots, r_{k-1} > = < b, Ab, \cdots, A^{k-1} b > .$$

[20]

Show also that

$$
\begin{aligned}
r_k^T r_j &= 0 \quad \text{for all} \quad j < k \quad \text{and} \\
p_k^T A p_j &= 0 \quad \text{for all} \quad j < k.
\end{aligned}
$$

[40]

(a) i. Suppose that the eigenvalue $\lambda_1$ of largest modulus and corresponding eigenvector $z_1$ of an $n \times n$ matrix $A$ have been computed by the Power method. Show that there is a non-singular matrix $S$, a product of an elementary permutation matrix and an elementary lower triangular matrix, such that

$$A = S \left[ \begin{array}{c|c} \lambda_1 & \gamma^T \\ \hline O & B \end{array} \right] S^{-1},$$

where $B$ is an $(n-1) \times (n-1)$ matrix and $\gamma$ is an $(n-1)$-column vector.

[25]

ii. Describe how the other eigenvalues and eigenvectors of $A$ could be computed. [20]

(b) It is given that the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

has an eigenvalue close to 3.4 and that a corresponding eigenvector approximately $(0.7, 1, 0.3)^T$. Obtain $2 \times 2$ matrix $B$ whose eigenvalues approximate the other eigenvalues of $A$. [30]