

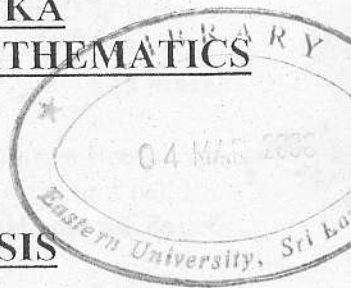
EASTERN UNIVERSITY, SRI LANKA
SPECIAL DEGREE EXAMINATION IN MATHEMATICS

(2004/2005)

MARCH/APRIL' 2007

PART II

MT 406 - FUNCTIONAL ANALYSIS



Answer all questions

Time: Three hours

1. a) Let $\{x_1, x_2, x_3, \dots, x_n\}$ be a linearly independent set of vectors in a normed linear space X . Prove that there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \alpha_2, \dots, \alpha_n$

$$\| \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n \| \geq c (|\alpha_1| + |\alpha_2| + \dots + |\alpha_n|).$$

[60 Marks]

- b) Prove that every finite dimensional subspace Y of a normed linear space is complete.

[40 Marks]

2. a) Let X, Y and Z be normed linear spaces. Prove that a linear operator $T: X \rightarrow Y$ is continuous if and only if there is a number $K \geq 0$ such that

$$\| Tx \| \leq K \| x \| \quad (x \in X).$$

[40 Marks]

- b) Define the norm, $\| T \|$ of a bounded linear operator $T: X \rightarrow Y$.

[10 Marks]

- c) Let $B[X, Y]$ be the vector space of all bounded linear operators of X into Y . Show that $\| \cdot \|$ as defined in (b) is a norm on $B[X, Y]$. (You may assume that $B[X, Y]$ is a vector space.)

[25 Marks]

- d) Let $S \in B[X, Y]$ and $T \in B[Y, Z]$. Let $T \circ S$ be the linear operator defined by

$$(T \circ S)(x) = T(S(x)) \quad (x \in X).$$

Show that $T \circ S$ is a bounded linear operator with

$$\| T \circ S \| \leq \| T \| \| S \|.$$

[15 Marks]

e) If $T \in B[X, Y]$ and $S \in B[Y, X]$ are such that $T \circ S$ is the identity operator in Y , what relation can you deduce between $\|T\|$ and $\|S\|$?

[10 Marks]

3. a) State the Hahn Banach Theorem for (real and complex) normed linear spaces.

[15 Marks]

b) Prove the Hahn Banach Theorem for complex normed spaces, assuming that it holds for real normed spaces.

[55 Marks]

c) Let X be a normed linear space. Use the Hahn Banach theorem to show that for every $x \in X$,

$$\|x\| = \text{Sup} \{ |f(x)| : f \in X^*, \|f\| \leq 1 \}$$

[30 Marks]

4. a) Let X, Y be normed linear spaces, and (T_n) be a sequence of bounded linear operators of X into Y . What does it mean to say that (T_n) is

- uniformly bounded;
- point wise bounded?

[15 Marks]

b) State and prove the Uniform Boundedness Theorem on the relation between two concepts defined in part(a) under suitable conditions. (You may assume the Baire's Category theorem.)

[45 Marks]

c) Let $X = Y = \text{Coo}$, the space of all eventually zero sequences

$x = (x_1, x_2, x_3, \dots)$ with the supremum norm

$$\|x\| = \text{Sup} \{ |x_i| : i = 1, 2, 3, \dots \}.$$

Show that the sequence of linear operators (T_n) defined by

$$T_n(x) = \{ x_1, 2x_2, 3x_3, \dots, nx_n, 0, 0, \dots \} \quad (x \in \text{Coo})$$

are pointwise bounded but not uniformly bounded. Why does this not contradict the Uniform Boundedness theorem?

[45 Marks]



5. a) State the Open Mapping theorem.

[15 Marks]

b) Let T be bounded linear operator from a Banach space X onto a Banach space Y . Prove that T has the property that the image $T(B_0)$ of the open unit ball $B_0 = B(0,1) \subset X$ contains an open ball about $0 \in Y$. (You may assume the Baire's Category theorem)

[85 Marks]

6. a) Let H be a Hilbert space. Prove that every bounded linear functional f on H can be represented in terms of the inner product namely,

$$f(x) = \langle x, z \rangle \quad (x \in H.)$$

where $z \in H$ depends on f . Further show that z is uniquely determined by f and has the norm $\|z\| = \|f\|$.

[60 Marks]

b) Define what is meant by a normed linear space is separable.

Prove with the usual notations that the sequence space l^p with $1 \leq p < \infty$ is separable.

[40 Marks]