EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION

IN MATHEMATICS, (2004/2005)

(MARCH/APRIL, 2007)

PART II

MT410 - NUMERICAL LINEAR ALGEBRA

Answer all Questions Time allowed: Three hours

1. (a) Define the term "positive definite" as applied to an $n \times n$ Hermitian matrix.

   (b) Prove that a Hermitian positive definite matrix A can be uniquely expressed as $A = LU$, where L is a unit lower-triangular matrix and U is an upper-triangular matrix.

   (c) Show that a Hermitian matrix A is positive definite if and only if $A = GG^H$, where G is a non-singular lower-triangular matrix. Determine G such that

$$GG^H = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

2. (a) An $n \times n$ elementary Hermitian matrix $H(\omega)$ is of the form

$$H(\omega) = I - 2\omega\omega^H, \quad \omega^H\omega = 1 \quad \text{or} \quad \omega = 0,$$

   where ω is an n- column vector and $\omega^H = \overline{\omega}^T$. Show that

$$[H(\omega)]^{-1} = H(\omega)$$

and that any product of elementary Hermitian matrices of the same order is unitary.

(b) Show that, for any $x \in \mathbb{R}^n$, there is an $n \times n$ real elementary Hermitian matrix $H(\omega)$ such that $H(\omega)x = ce_1$ where $c^2 = x^T x$ and $e_1 = (1, 0, 0, \cdots, 0)^T \in \mathbb{R}^n$.

Explain the optimal choice of the sign of c for the computation of ω.

(c) Find an upper triangular matrix U such that $HA = U$, where H is a product of elementary Hermitian matrices and

$$A = \begin{bmatrix} 1 & 6 & -1 \\ 2 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix},$$

making the optimal choice of sign in each stage of the process. Hence solve $Ax = e_1$, where $e_1 = (1, 0, 0)^T$.

3. (a) Define the phrase " strictly diagonally dominant" as applied to an $n \times n$ matrix.

(b) Let $A = I - L - U$ be an strictly diagonally dominant, where I is the $n \times n$ identity matrix, L a strictly lower-triangular matrix and U a strictly upper-triangular matrix. Prove that, for arbitrary $x^{(0)}$, the sequence of vectors $\{x^{(r)}\}$ defined by

$$x^{(r+1)} = (I - L)^{-1}[Ux^{(r)} + b], \qquad r = 0, 1, 2, \cdots,$$

converge to x, where $Ax = b$. Prove also that, for some corresponding vector and matrix norms,

$$\|x^{(r+1)} - x\| \leq \frac{\|(I - L)^{-1}U\|}{1 - \|(I - L)^{-1}U\|}\|x^{(r+1)} - x^{(r)}\|, \quad r = 0, 1, 2, \cdots$$

(c) The following equations are to be solved by Gauss-Seidal iteration:

$$\begin{array}{rcrcrcrcl} 10x_1 & & & + & x_3 & + & x_4 & = & 2, \\ x_1 & + & x_2 & & & + & 10x_4 & = & 2, \\ & & 5x_2 & & & + & x_4 & = & 1, \\ x_1 & & & + & 5x_3 & & & = & 1. \end{array}$$

Starting with $x^{(0)} = 0$, obtain $x^{(1)}$, $x^{(2)}$ and bound for $\|x - x^{(2)}\|_\infty$.

4. (a) Define the terms " Upper Hessenberg" and " Tridiagonal" as applied to an $n \times n$ matrix A.

Show that there exists a unitary matrix S, a product of elementary Hermitian matrices, such that $S^H AS$ is an upper Hessenberg matrix.

(b) Determine a tridiagonal matrix T such that $S^H AS = T$, where S is unitary and

$$A = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 3 & 3 & 4 \\ 4 & 3 & 3 & 4 \\ 0 & 4 & 4 & -3 \end{bmatrix}.$$

Choose an appropriate sign for the construction of each elementary Hermitian matrix needed.

5. (a) Let A be an $n \times n$ Hermitian positive definite matrix with eigenvectors u_i corresponding eigenvalues λ_i that satisfy

$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0.$

Let

$$\sigma_r x^{(r+1)} = Ax^{(r)}, \quad r = 0, 1, 2, \cdots, \tag{1}$$

where σ_r is a component of $Ax^{(r)}$ of largest modulus. Given that $x^0 = \alpha_1 u_1 + \alpha_2 u_2 \cdots + \alpha_n u_n$ with $\alpha_1 \neq 0$, show that the sequence $\{x^{(r)}\}$ converges to the subspace spanned by u_1 and that the sequence $\{|\sigma_r|\}$ converges to λ_1.

(b) Let

$$\beta_r = \frac{\sigma_r x^{(r)H} x^{(r+1)}}{x^{(r)H} x^{(r)}}, \quad r = 0, 1, 2, \cdots .$$

Show that $\{\beta_r\}$ converges to λ_1.

(c) Starting with $x^{(0)} = (1, 1, 0)^T$, obtain $x^{(1)}, x^{(2)}, x^{(3)}$ by applying (1) to the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Hence calculate β_2.

6. (a) Suppose that the eigenvalue λ_1 of largest modulus and corresponding eigenvector z_1 of an $n \times n$ matrix A have been computed by the Power method.

i. Show that there is a non-singular matrix S, a product of an elementary permutation matrix and elementary lower triangular matrix, such that

$$A = S \left[\begin{array}{c|c} \lambda_1 & \gamma^T \\ \hline 0 & B \end{array} \right] S^{-1},$$

where B is an $(n-1) \times (n-1)$ matrix and γ is an $(n-1)$-column vector.

ii. Describe how the other eigenvalues and eigenvectors of A could be computed

(b) It is given that the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

has an eigenvalue close to 3.4 and a corresponding eigenvector approximately $(0.7, 1, 0.3)^T$. Obtain 2×2 matrix B whose eigenvalues approximate the other eigenvalues of A.