EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE 2001/2002

(April'2002)

FIRST SEMESTER

Repeat

MT 103 - VECTOR ALGEBRA & CLASSICAL MECH.

Answer all questions

Time : Three hours

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1. (a) For any three vectors $\underline{a}, \underline{b}, \underline{c}$, prove that the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}.$$

Hence prove that

$$(\underline{a} \wedge \underline{b}) \cdot (\underline{c} \wedge \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c}).$$

(b) Let $\underline{l}, \underline{m}$ and $\underline{n}$ be three non-zero and non-coplanar vectors such that any two of them are not parallel. By considering the vector product $(\underline{r} \wedge \underline{l}) \wedge (\underline{m} \wedge \underline{n})$, prove that any vector $\underline{r}$ can be expressed in the form

$$\underline{r} = (\underline{r} \cdot \underline{\alpha})\underline{l} + (\underline{r} \cdot \underline{\beta})\underline{m} + (\underline{r} \cdot \underline{\gamma})\underline{n}.$$

Find the vectors $\underline{\alpha}, \underline{\beta}, \underline{\gamma}$ in terms of $\underline{l}, \underline{m}, \underline{n}$.

(c) Let $\underline{p}, \underline{q}$ and $\underline{r}$ be three non-zero vectors such that $\underline{r} - (\underline{p} \wedge \underline{q}) = \alpha \underline{q}$ and $\underline{p} \cdot \underline{q} = 0$ where α is a scalar. Show that

$$\underline{p} = \underline{q} \wedge \frac{\underline{r}}{|\,\underline{q}\,|^2} \quad \text{and} \quad \alpha = \frac{\underline{q} \cdot \underline{r}}{|\,\underline{q}\,|^2}.$$

2. Define the terms "conservative vector field" and "scalar potential".

If the force field $\underline{F} = \underline{\nabla}\phi$, where ϕ is a single-valued and has continuous partial derivative, show that the work done by moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ to another point $P_2 \equiv (x_2, y_2, z_2)$ in this field is independent of its path joining the two points.

Conversely, if $\int_c \underline{F} \cdot d\underline{r}$ is independent of the path c joining any two points, show that there exists a scalar function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

Show that the field

$$\underline{F} = (2xy + z^3)\underline{i} + x^2\underline{j} + 3xz^2\underline{k}$$

is a conservative force field. Find the scalar potential ϕ such that $\underline{F} = \underline{\nabla}\phi$.

Hence find the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$.

3. (a) State and prove Green's theorem on the plane.

Verify Green's theorem in the plene for

$$\oint_C (xy + y^2)dx + x^2 dy,$$

where C is the closed boundary of the region defined by $y = x$, $y = x^2$.

(b) State the divergence theorem and use it to evaluate $\int\int_S \underline{A} \cdot \underline{n}\, dS$, where $\underline{A} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1$ and $z = 0, z = 1$.

4. Show that the tangential and normal acceleration of a particle which describes the plane curve $S = f(\psi)$ are $\frac{dv}{dt}$ and $\frac{v^2}{\rho}$ respectively, where v is the speed of the particle and ρ is the radius of the curvature.

A particle slides on a rough wire in the form of cycloid $s = 4a \sin\psi$ which is fixed in a vertical plane with its axis vertical and vertex downwards. The particle is projected from the vertex $\psi = 0$ with speed u so that it comes to rest at the cusp $(\psi = \frac{\pi}{2})$. Show that

$$e^{\mu\pi} = \mu^2 + (\mu^2 + 1)\left(\frac{u^2}{4ag}\right),$$

where μ is the coefficient of friction between the particle and the wire. Show also that, when the particle slides from rest at the cusp, it will come to rest at the vertex if $e^{-\mu\pi} = \mu^2$.

5. A smooth hollow right circular cone is placed with its vertex downwards and the axis vertical. A particle is projected horizontally along the inner surface of the cone with speed $\sqrt{\dfrac{2gh}{n^2+n}}$ at height h above the vertex. Show that the lowest point of its path will be at height $\dfrac{h}{n}$ above the vertex. Find also the reaction of the cone on the particle at this lowest point.

6. Establish the equation $\underline{F}(t) = m(t)\dfrac{d\underline{v}}{dt} + \lambda \underline{u}$ for the motion of a rocket of varying mass $m(t)$ moving in a straight line with velocity $\underline{u}$ under a force $\underline{F}(t)$, matter being emitted at a constant rate λ with a velocity $\underline{u}$ relative to the rocket.

A rocket with initial mass M is fired upwards. Matter is ejected with relative velocity u at a constant rate eM. M' being the mass of the rocket without fuel. Show that the rocket can't rise at once unless $eu > g$.

If it just rises vertically at once, show that its greatest velocity is

$$u \ln \frac{M}{M'} - \frac{g}{e}\left(1 - \frac{M'}{M}\right)$$

and the greatest height reached is

$$\frac{u^2}{2g}\left\{\ln\left(\frac{M}{M'}\right)\right\}^2 + \frac{u}{e}\left(1 - \frac{M'}{M} - \ln\frac{M}{M'}\right).$$