EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE -2008/2009

FIRST SEMESTER (Feb., 2010)

MT 302 - ANALYSIS IV(COMPLEX ANALYSIS)

(Proper)

Answer all questions — Time: Three hours

1. (a) Define what is meant by a complex-valued function f, defined on a domain $D(\subseteq \mathbb{C})$, has a limit at $z_0 \in D$.

   i. Prove that if a complex-valued function f has a limit at $z_0 \in D$, then it is unique.

   ii. Show that
   $$\lim_{z \longrightarrow 3i} \frac{z^2+6-iz}{z-3i} = 5i.$$

   (b) i. Let $f : S \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ and let z_0 be an interior point of S. Define what is meant by f being continuous at z_0 and on S.

   Show that the function
   $$f(z) = z^2$$
   is continuous at $z = z_0$.

   ii. Show that
   $$\left|\exp(z^2)\right| \leq \exp\left(|z|^2\right).$$

2. (a) i. Let $f : A \longrightarrow \mathbb{C}$ and $A \subseteq \mathbb{C}$ be an open set . Define what is meant by f being analytic at $z_0 \in A$.

ii. Show that if $z = x + iy$ and a function $f(z) = U(x, y) + iV(x, y)$ is analytic at $z_0 = x_0 + iy_0$, then the equations

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \text{ and } \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

are satisfied at every point of some neighborhood of z_0.

iii. Show that the function

$$f(z) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} + \dfrac{y^3 - x^3}{x^2 + y^2}, & \text{for } \quad x^2 + y^2 \neq 0; \\ 0, & \text{for } \quad x^2 + y^2 = 0. \end{cases}$$

does not have derivative at $z = 0$.

(b) i. Show that the function $U(x, y) = e^{-x}(x \sin y - y \cos y)$ is harmonic.

ii. Find a function $V(x, y)$ such that $f(z) = U(x, y) + iV(x, y)$ is analytic.

3. (a) Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_0 is any point inside C then the n^{th} derivative of f at $z = z_0$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad \text{where } n = 0, 1, 2, 3, ...$$

Prove the above result for $n = 0$.

Hence prove that if $f(z)$ is analytic inside and on a circle C of radius r and center at $z = a$,

$$|f^n(a)| \leq \frac{M.n!}{r^n} \qquad n = 0, 1, 2, 3, ...,$$

where M is a constant such that $|f(z)| < M$.

(b) Using the above result prove the Liouville's theorem for bounded functions.

(c) Show that

$$\int_C \frac{dz}{z+1} = 2\pi i \quad \text{if C is the circle , } C : |z| = 2.$$

4. (a) i. Define what is meant by a path $\gamma : [\alpha, \beta] \longrightarrow \mathbb{C}$.

ii. For a path γ and a continuous function $f : \gamma \longrightarrow \mathbb{C}$, define $\int_\gamma f(z)dz$.

(b) Prove that if $w(t)$ is a continuous complex - valued function of t such that $a \leq t \leq b$, then

$$\int_a^b w(t)dt \leq \int_a^b | w(t) | \, dt.$$

(c) State and prove the Taylor's theorem.

Show that

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}; \quad |z| < \infty.$$

5. (a) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \to \mathbb{C}$, where $D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}$.

Define what is meant by

i. f having a singularity at z_0;

ii. the order of f at z_0;

iii. f having a pole or zero at z_0 of order m;

iv. f having a simple pole or simple zero at z_0.

(b) Prove that $\text{ord}(f, z_0) = m$ if and only if

$$f(z) = (z - z_0)^m g(z), \quad \forall z \in D^*(z_0; \delta),$$

for some $\delta > 0$, where g is analytic in $D^*(z_0; \delta) := \{z : |z - z_0| < \delta\}$ and $g(z_0) \neq 0$.

(c) Prove that if f has a simple pole at z_0, then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z_0),$$

where $\text{Res}(f; z_0)$ denotes the residue of $f(z)$ at $z = z_0$.

6. (a) Let f be analytic in $\{z \ : \ Im(z) \geq 0\}$, except possibly for finitely many singularities, none of them on the real axis. Suppose there exist $M, \ R \ > 0$ and $\alpha > 1$ such that

$$|f(z)| \leq \frac{M}{|z|^\alpha}, \quad |z| \geq R \quad \text{with} \quad Im(z) \geq 0.$$

Prove that $I = \int_{-\infty}^{\infty} f(x)dx$ converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues in the upper half plane.}$$

(b) A function $\phi(z)$ is zero when $z = 0$, and is real when z is real, and is analytic when $|z| \leq 1$. If $f(x, y)$ is the imaginary part of $\phi(x + iy)$, then prove that

$$\int_0^{2\pi} \frac{x \sin\theta}{1 - 2x\cos\theta + x^2} f(\cos\theta, \sin\theta)d\theta = \pi\phi(x) \text{ holdes when } -1 < x < 1.$$