EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE 2002/2003

June/July, 2003

FIRST SEMESTER

REPEAT

MT 201 - VECTOR SPACES AND MATRICES

Answer all questions | Time: Three hours

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1. (a) Explain what is meant by

   i. a vector space,

   ii. a subspace of a vector space.

   [20 marks]

   (b) Let V be a vector space over a field $\mathbb{F}$ and W be a non-empty subset of V. Prove that W is a subspace of V if and only if $ax + by \in W$ for every $x, y \in W$ and for every $a, b \in \mathbb{F}$.

   [35 marks]

   (c) Let $M_{m\times n}$ be the set of all real $m \times n$ matrices. For any two matrices $A = [\alpha_{ij}]_{m\times n}$ and $B = [\beta_{ij}]_{m\times n}$ in $M_{m\times n}$, and for any $\lambda \in \mathbb{R}$ define an addition $\oplus$ and scalar multiplication $\odot$ as follows:

$$[\alpha_{ij}]_{m\times n} \oplus [\beta_{ij}]_{m\times n} = [\alpha_{ij} + \beta_{ij}]_{m\times n},$$

$$\lambda \odot [\alpha_{ij}]_{m\times n} = [\lambda \alpha_{ij}]_{m\times n}.$$

   Prove that $(M_{m\times n}, \oplus, \odot)$ is a vector space over the field $\mathbb{R}$. [45 marks]

2. (a) Define the following:

(i) a linearly independent set of vectors,

(ii) a basis for a vector space,

(iii) dimension of a vector space. [15 marks]

(b) Let V be an $n-$dimensional vector space. Prove the following:

(i) A linearly independent set of vectors of V with n elements is a basis for V;

(ii) Any linearly independent set of vectors of V may be extended to a basis for V.

[50 marks]

(c) i. Let V be a vector space over the field $\mathbb{R}$. Suppose that (x, y, z) is linearly independent sequence of vectors in V. Let $u = x - y$, $v = y - z$, $w = z + \alpha x$, where α is a scalar. Prove that the sequence (u, v, w) is linearly dependent if and only if $\alpha = -1$.

ii. In a vector space the sequence of vectors $(x_1, x_2, \cdots, x_n)$ is given to be linearly independent and $y = \sum_{i=1}^{n} \alpha_i x_i$ where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are scalars with $\alpha_1 \neq 0$. Prove that the sequence $(y, x_2, x_3, \cdots, x_n)$ is also linearly independent.

[20 marks]

(d) Extend the subset $\{(1, -2, 5, -3), (0, 7, -9, 2)\}$ to a basis for $\mathbb{R}^4$.

[15 marks]

3. (a) State and prove the dimension theorem for two subspaces of a finite dimensional vector space. [30 marks]

(b) Let $\{u_1, u_2, ..., u_n\}$ be a basis of a finite $n-$dimensional vector space V.

i. Prove that for each s in the range 1 to $n - 1$, inclusive,

$$V = \langle\{u_1, u_2, \cdots, u_s\}\rangle \oplus \langle\{u_{s+1}, u_{s+2}, \cdots, u_n\}\rangle.$$

ii. Show that if S_1 and S_2 are two direct complements of S in V, then

$$\dim V - 2\dim S \leq \dim(S_1 \cap S_2) \leq \dim V - \dim S.$$

[45 marks]

(c) Let V be a finite dimensional vector space and W be a subspace of V. Prove that the quotient space V/W is also finite dimensional and $\dim(V/W) = \dim V - \dim W$. [25 marks]

4. (a) Define

(i) Range space $R(T)$,

(ii) Null space $N(T)$

of a linear transformation T from a vector space V into another vector space W.

[20 marks]

Find $R(T)$, $N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z), \ \forall\ (x, y, z) \in \mathbb{R}^3.$$

Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation.

[30 marks]

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by
$T(x, y, z) = (2x + y + 3z, 3x - y + z, -4x + 3y + z), \ \forall\ (x, y, z) \in \mathbb{R}^3.$
Let $B_1 = \{(1,1,0), (0,1,1), (1,0,1)\}$ and $B_2 = \{(1,1,1), (1,2,3), (2,-1,1)\}$ be bases for $\mathbb{R}^3$. Find

(i) the matrix representation of T with respect to the basis B_1;

(ii) the matrix representation of T with respect to the basis B_2 by using the transition matrix.

[50 marks]

5. (a) Define the following terms as applied to a matrix:

i. Rank,

ii. Echelon form,

iii. Row reduced echelon form.

[15 marks]

(b) Let A be an $n \times n$ matrix. Prove that

i. row rank of A is equal to column rank of A;

ii. if B is an $n \times n$ matrix, obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

[45 marks]

(c) Find the rank of the matrix

$$\begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}.$$

[20 marks]

(d) Find the row reduced echelon form of the matrix

$$\begin{bmatrix} -1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}.$$

[20 marks]

6\. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

[10 marks]

Reduce the augmented matrix of the following system of linear equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

to its row reduced echelon form and hence determine the conditions on a, b, c, d, e and f such that the system has

(i) a unique solution;

(ii) no solution;

(iii) more than one solution.

[30 marks]

(b) The system of equations

$$\begin{aligned} 2x + 3y + z &= 5 \\ 3x + 2y - 4z + 7t &= k + 4 \\ x + y - z + 2t &= k - 1 \end{aligned}$$

is known to be consistent. Find the value of k and general solution of the system.

[30 marks]

(c) State Crammer's rule for 3×3 matrix and use it to solvethe following system of equations

$$\begin{aligned} 2x_1 - 5x_2 + 2x_3 &= 7 \\ x_1 + 2x_2 - 4x_3 &= 3 \\ 3x_1 - 4x_2 - 6x_3 &= 5. \end{aligned}$$

[30 marks]