EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE 2002/2003

FIRST SEMESTER

(June/July, 2003)

MT306 - PROBABILITY THEORY

Answer all questions

Time : Two hours

1. (a) Let Y be a negative binomial random variable with parameters r and p and its probability mass function be given by,

$$P(Y=y) = \begin{cases} \begin{pmatrix} y-1 \\ r-1 \end{pmatrix} p^r q^{y-r}; & y = r, r+1, r+2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Find,

i. the expected value of Y,

ii. the variance of Y,

iii. the moment generating function of Y.

(b) The mean muscular endurance score of a random sample of 60 subjects was found to be 145 with standard deviation of 40. Construct a 95% confidence interval for the true mean. Assume the sample size to be large enough for normal approximation. What size of sample is required to estimate the mean within 5 of the true mean with a 95% confidence?

2. A particular fast-food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and leaving the service window, and Y_2, the time that a customer waits in line before reaching the service window. Because Y_1 contains the time a customer waits in line, we must have $Y_1 \geq Y_2$, The relative frequency distribution of observed values of Y_1 and Y_2 can be modelled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1} ; & 0 \leq y_2 \leq y_1 < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(Y_1 < 2, Y_2 > 1)$

(b) find $P(Y_1 \geq 2Y_2)$

(c) If 2 minutes elapse between a customer's arrival at the store and departure from the service window, find the probability that he waited in line less than 1 minute to reach the window.

(d) Are Y_1 and Y_2 independent?

(e) The random variable $Y_1 - Y_2$ represents the time spent at the service window. Find $E(Y_1 - Y_2)$ and $V(Y_1 - Y_2)$. Is it highly likely that a customer would spend more than 2 minutes at the service window?

3. (a) State the Cramer-Rao inequality

(b) Given the probability density function,

$$f(x,\theta) = [\pi\{1+(x-\theta)^2\}]^{-1} \quad ; \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

show that the Cramer- Rao lower bound of variance of an unbiased estimator of θ is $\frac{2}{n}$, where n is the size of the random sample from this distribution.

4. (a) A random sample $X_1, X_2, \cdots, X_n$ is obtained from a distribution with probability density function,

$$f(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad ; \quad 0 \leq x < \infty,$$

where α and β are unknown parameters. Estimate α and β by using the the method of moments.

(b) Show that if X is a random variable having the Poisson distribution with the parameter λ and $\lambda \to \infty$, then the moment generating function of $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ approaches the moment generating function of the standard normal distribution.

(c) Determine the maximum likelihood estimators of the parameters of the following distributions:

i. Geometric population with parameter p.

ii. Exponential population with parameter θ