EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE (2000/2001)

(MAY' 2001)

FIRST SEMESTER

MT 201 - VECTOR SPACES AND MATRICES

Answer all questions **Time : Three hours**

1. (a) Define what is meant by:

   i. a vector space;

   ii. a subspace of a vector space.

   Let V be a vector space over the field F and W be a non-empty subset of V. Prove that W is a subspace of V if and only if $ax + by \in W$ for every $x, y \in W$ and for every $a, b \in F$.

   (b) Let $V = \{f \mid f : \Re \to \Re, \quad f(x) > 0 \quad \text{for all } x \in \Re\}$. For any $f, g \in V$ and for any $r \in \Re$, define an addition $\oplus$ and a scalar multiplication $\odot$ as follows:

$$(f \oplus g)(x) = f(x).g(x) \quad \text{for every } x \in \Re;$$

$$(r \odot f)(x) = [f(x)]^r \quad \text{for every } x \in \Re.$$

   Show that $(V, \oplus, \odot)$ forms a vector space over $\Re$.

(c) Let $V = \Re^3$ be a vector space over the field $\Re$, which of the following are subspace of V;

i. $W_1 = \{(a, b, 0); \ a, b \in \Re\}$,

ii. $W_2 = \{(a, b, c); \ a, b, c \in Q\}$,

iii. $W_3 = \{(a, b, c); \ a + b + c = 0\}$.

2. (a) Define the following:

i. a linearly independent set of vectors;

ii. a basis for a vector space;

iii. dimension of a vector space.

(b) Let V be an $n-$ dimensional vector space.

Prove the following:

i. A linearly independent set of vectors of V with n elements is a basis for V.

ii. Any linearly independent set of vectors of V may be extended as a basis for V.

iii. If L is a subspace of V, then there exists a subspace M of V such that $V = L \oplus M$.

(c) Extend the subset $\{(1, 2, 1), (3, -4, 7)\}$ to a basis for $\Re^3$.

3. (a) State the Dimension theorem for two subspaces of a finite dimensional vector space.

(b) Let W_1, W_2 and W_3 be subspaces of a finite dimensional vector space V.

Show that,

$$\dim(W_1+W_2+W_3) \leq \dim W_1 + \dim W_2 + \dim W_3 - \dim(W_1 \cap W_2) - \dim(W_2 \cap W_3) - \dim(W_1 \cap W_3) + \dim(W_1 \cap W_2 \cap W_3).$$

(c) If $W_1 = \langle\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}\rangle$ and $W_2 = \langle\{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}\rangle$ are subspaces of $\Re^4$. Find

i. $\dim W_1$;

ii. $\dim W_2$;

iii. $\dim(W_1 \cap W_2)$;

iv. $\dim(W_1 + W_2)$.

Verify the dimension theorem.

4. (a) Define

i. Range space $R(T)$,

ii. Null space $N(T)$,

of a linear transformation T from a vector space V in to another vector space W.

Find $R(T)$ and $N(T)$ of the linear transformation of $\Re^3$, defined by $T(x,y,z) = (x+2y+3z,\ x-y+z,\ x+5y+5z)$.

Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation.

(b) Let $\phi : \Re^3 \rightarrow \Re^3$ be a linear transformation defined by, $\phi(x, y, z) = (x,\ x + y,\ y - z)$ and let $B_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $B_2 = \{(1, 1, 0), (-1, 1, 0), (0, 0, 1)\}$ be bases for $\Re^3$. Find

i. the matrix representation of ϕ with respect to the basis B_1;

ii. the matrix representation of ϕ with respect to the basis B_2 by using the transition matrix;

iii. the matrix representation of ϕ with respect to the basis B_2 directly.

5. (a) Define the following terms as applied to a square matrix:

i. Minor,

ii. Co-factor,

iii. Adjoint.

(b) Let A be any $n \times n$ matrix. If E is an $n \times n$ elementary matrix, show that $\det(EA) = \det E \det A$. Hence or otherwise prove that if A and B are two $n \times n$ square matrices, then $\det(AB) = \det A \det B$.

(c) Let A and B be two $n \times n$ non - singular matrices.

Prove the following:

i. $\operatorname{adj}(\lambda A) = \lambda^{n-1} \operatorname{adj} A$ for all real number λ;

ii. $\operatorname{adj}(A^{-1}) = (\operatorname{adj} A)^{-1}$;

iii. $\det(\operatorname{adj} A) = (\det A)^{n-1}$;

iv. $\operatorname{adj}(\operatorname{adj} A) = (\det A)^{n-2} A$;

v. $\operatorname{adj}(\operatorname{adj}(\operatorname{adj} A)) = (\det A)^{n^2-3n+3} A^{-1}$.

State any results (theorem) that you use to prove the above results.

(d) Find the adjoint of the following matrix A and hence find it's inverse,

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

6. (a) State the necessary and sufficient condition for a system of linear equations to be consistant.

Reduce the augmented matrix of the following system of linear equations to it's row reduced echelon form and hence determine the values of k such that the system has;

i. a unique solution

ii. no solution

iii. more than one solution

$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1. \end{aligned}$$

(b) State and prove **Crammer's** rule for 3×3 matrix and use it to solve:

$$\begin{aligned} x_1 + 2x_2 - cx_3 &= -4 \\ 3x_1 + 5x_2 - cx_3 &= -5 \\ -2x_1 - x_2 - cx_3 &= -5. \end{aligned}$$

(c) Prove that the system,

$$\begin{aligned} 2x + 3y - 2z &= 5 \\ x - 2y + 3z &= 2 \\ 4x - y + 4z &= 1 \end{aligned}$$

is inconsistant.